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Some Domain Decompostion Algorithms for Nonselfadjoint Elliptic and Parabolic Partial Differential Equations

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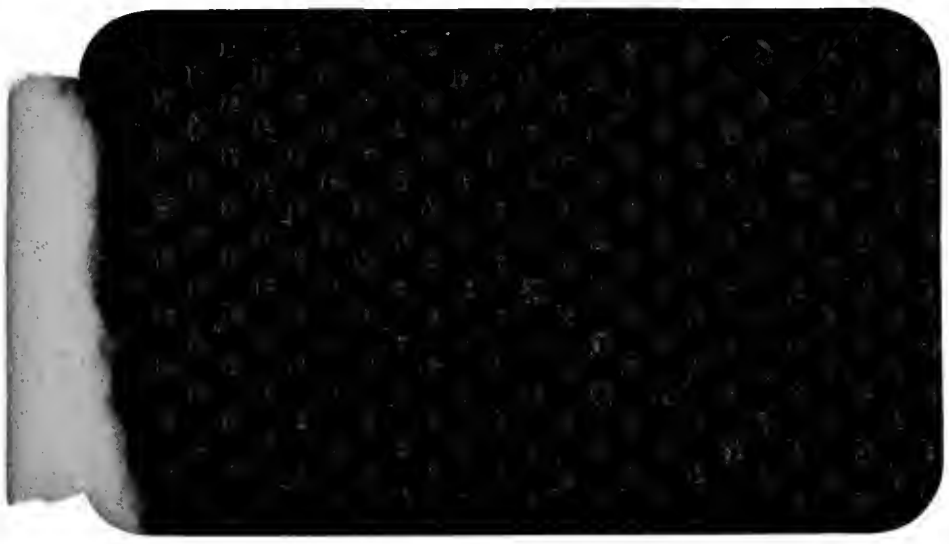

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Abstract

In this dissertation, we consider the solution of linear systems of algebraic equations that arise from finite element problems. We study some additive Schwarz type domain decomposition algorithms for general, not necessarily selfadjoint, linear elliptic and parabolic PDEs. We use the GMRES method to solve the resulting linear systems of equations. In each step of the iteration, a number of smaller linear systems, which correspond to the restriction of the original problem to subregions, are solved instead of the large system. The number of subproblems can be potentially large and these methods are therefore promising for parallel computation. We estimate the rate of convergence of the algorithms. Numerical experiments are also reported.

Contents

Chapter 1

Introduction

1.1 An overview

In this thesis, we consider the solution of linear systems of algebraic equations that arise from finite element problems. We study the additive Schwarz method and the iterative substructuring method for general, not necessarily selfadjoint, linear, second order, elliptic and parabolic partial differential equations.

Domain decomposition techniques are very powerful iterative methods for such problems. In each step of an iteration, a number of smaller linear systems, which correspond to the restriction of the original problem to subregions, are solved instead of the large original system. The number of subproblems can be potentially large and these methods therefore are promising for parallel computation. These are divide and conquer methods and the central mathematical question is to obtain a bound on the rate of convergence of the iteration. Borrowing a term from structural engineering computations, the subregions are often called substructures. We thus have two partitions of the region into substructures, which sometimes is used to define a coarse, global model, and into elements of the finite element model.

The iterative methods most commonly used are the conjugate gradient method for the symmetric, positive definite case and the generalized conjugate residual methods (GMRES) for the general, nonsymmetric case. If the symmetric part of the operator is positive definite, with respect to a suitable inner product, convergence can be guaranteed. In this thesis, the rate of convergence of all algorithms will be estimated. We show that the additive Schwarz algorithm is optimal for both elliptic and parabolic problems in R^2 and R^3 in the sense that the rate of convergence is independent of both the coarse mesh size, defined by the substructures, and the fine mesh size. The iterative substructuring algorithm is not optimal in the above sense, however, in the R^2 case the corresponding rate of convergence depends only mildly on the mesh parameters. A modified additive Schwarz algorithm is also introduced for parabolic problems in R^2. The rate of convergence is independent of the fine mesh size.

So far, most work on domain decomposition methods has concentrated on selfadjoint elliptic equations. Several optimal algorithms are known for this case, see [4], [5], [2], [13], [22], etc. For previous work on nonselfadjoint elliptic problems, see [8], [20], [24].

The numerical schemes for the nonselfadjoint problems fall into one of the following two categories. The first class of methods consists of higher order accurate methods such as the Galerkin's method, which is the one we will concentrate on, and of the finite difference methods based on using centered difference approximations for the convection terms. These methods can produce very good solutions in the situation where either the original equation has a smooth solution or the mesh size is relatively small compared with the Reynold's number of the equation, roughly defined as the ratio of the magnitude of the convection coefficient and the minimal

eigenvalue of the diffusion coefficient matrix. A discussion can be found in [25]. The disadvantage of these methods is that they can produce large oscillation when the solution is discontinuous and the mesh size is relatively large. In the other class we find the streamline diffusion methods, see [19], which produce nonoscillating solutions when the Galerkin's method does not work well. In this thesis, we consider only the Galerkin's method.

This thesis is organized as follows. In the remainder of this chapter, we first present some basic Sobolev spaces and the GMRES method, which is the main iterative method we shall use for solving our linear system of equations. Then, we develop an abstract theory for the additive Schwarz method, which unifies a number of separate results developed for some different domain decomposition algorithms in recent years. In Chapter 2, we apply our abstract theory to the additive Schwarz method for the stationary convection-diffusion problems in R^2 and R^3 and an iterative substructuring method for the same class of problems in R^2. We devote Chapter 3 to the parabolic convection-diffusion problems. Besides the two algorithms just mentioned, we also introduce a modified additive Schwarz method, which is designed for parabolic problems in R^2. Finally, in Chapter 4, we present some results of our numerical experiments.

1.2 Preliminary materials

1.2.1 Some Sobolev spaces

We will need a number of Sobolev spaces. Let Ω be an bounded open region in R^2 or R^3. By $L^2(\Omega)$, we denote the space of square integrable functions on Ω. $(\cdot,\cdot)$ denotes the $L^2(\Omega)$ inner product. For integer $m \geq 0$, $H^m(\Omega)$ is the subspace of $L^2(\Omega)$ for which,

$$\|u\|_{H^m(\Omega)} = (\int_\Omega \sum_{|\alpha|\leq m} |\ (\partial/\partial x)^\alpha u(x)\ |^2\ dx)^{1/2} < \infty. \tag{1.1}$$

For $s = m + \sigma > 0,\ 0 < \sigma < 1$, we define $H^s(\Omega)$ in terms of the norm,

$$\begin{aligned} \|u\|_{H^s(\Omega)} \ &= \ (\|u\|^2_{H^m(\Omega)} + \\ &\int_\Omega\int_\Omega \sum_{|\alpha|=m} |((\partial/\partial x)^\alpha u(x) - (\partial/\partial y)^\alpha u(y))(x-y)^{-(1+\sigma)}\ |^2 dxdy)^{1/2}. \end{aligned} \tag{1.2}$$

We will also need the seminorm $|\ u\ |_{H^s(\Omega)}$, which is obtained by dropping all the terms with derivatives of order less than m in (1.1) and the first term in (1.2).

We also use $H^s_0(\Omega),\ s > 0$, which is a subspace of $H^s(\Omega)$ defined as the closure in $H^s(\Omega)$ of $D(\Omega)$, the space of C^∞ functions with compact support in Ω.

1.2.2 The GMRES method

In Eisenstat, Elman and Schultz [14], the conjugate gradient method was generalized to solve nonsymmetric linear systems of equations. The so called generalized minimum residual method, GMRES, has been shown in practice to be very powerful for a large class of problems. In their paper the GMRES method and the corresponding theory in $L^2(\Omega)$ are given, but in fact this algorithm and its theory can be extended easily to any Hilbert spaces. We shall describe the algorithm and state the theory without proof.

Let P be an linear operator defined on the finite dimensional space R^n with the inner product $[\cdot,\cdot]$, which in practice is chosen to take advantage of some special properties of P. In our applications, P is not symmetric but positive definite with respect to $[\cdot,\cdot]$, i.e. there exist some $x,\ y \in R^n$,

such that

$$[Px, y] \neq [x, Py],$$

but there exists a constant $c > 0$, such that

$$[x, Px] \geq c[x, x], \quad \forall x \in R^n.$$

We are interested in using the GMRES method to solve the following linear system of equations on R^n:

$$Px = b,$$

where b is given in R^n.

The iteration begins with an initial approximate solution $x_0 \in R^n$ and an initial residual $r_0 = b - Px_0$. At the m^{th} iteration, a correction vector z_m is computed in the Krylov subspace

$$\mathcal{K}_m(r_0) = \text{span}\{r_0, Pr_0, \cdots, P^{m-1}r_0\},$$

which minimizes the following residual

$$\min_{z \in \mathcal{K}_m(r_0)} \| b - P(x_0 + z) \|,$$

where the norm is defined to be

$$\| \cdot \| = \sqrt{[\cdot, \cdot]}.$$

The m^{th} iterate is then $x_m = x_0 + z_m$. It can be shown that if we perform exact arithmetic operations, then the solution would be reached in no more than n iterations.

GMRES Algorithm:

Choose x_0

Compute $r_0 = b - Px_0$

Set $p_0 = r_0$

For $i = 0$ **Step** 1 **Until Convergence Do**

$a_i = [r_i, Pp_i]/[Pp_i, Pp_i]$

$x_{i+1} = x_i + a_i p_i$

$r_{i+1} = r_i - a_i P p_i$

$p_{i+1} = r_{i+1} + \sum_{j=0}^{i} b_j^{(i)} p_j$

where for $j \leq i$, $b_j^{(i)} = -[Pr_{i+1}, Pp_j]/[Pp_j, Pp_j]$.

The work and storage requirements per iteration becomes very expensive if a large number of iterations is needed to achieve a given accuracy. Therefore, to reduce the number of iteration becomes a very important issue for nonsymmetric problems. There are some alternative ways to save storage, such as the Orthomin(k) and the k-step restarted GMRES method, which will not be considered in the paper.

According to the theory of Eisenstat, Elman and Schultz, the rate of convergence of the GMRES method can be characterized by the ratio of the minimal eigenvalue of the symmetric part of the operator and the norm of the operator. Let us define those two quantities as follows:

$$c_p = \inf_{x \neq 0} \frac{[x, Px]}{[x, x]}$$

and

$$C_p = \sup_{x \neq 0} \frac{\| Px \|}{\| x \|}.$$

Let P^* be the adjoint operator of P with respect to $[\cdot, \cdot]$. Then the energy contributed by the symmetric part is

$$[\frac{P + P^*}{2} x, x] = [Px, x].$$

We have the following theorem for the rate of convergence. In the case that $[\cdot, \cdot] = (\cdot, \cdot)$, the proof is given by Eisenstat, Elman and Schultz [14].

Theorem 1.1 *If $c_p > 0$, then the GMRES method converges and at the m^{th} iteration, the residual is bounded as*

$$\| r_m \| \leq (1 - \frac{c_p^2}{C_p^2})^{m/2} \| r_0 \|.$$

1.3 The abstract theory for the additive Schwarz method

Let V be a Hilbert space consisting of real functions defined on $\Omega \subset R^d$, where d is the dimension, with an inner product $(u, v)_V$ and the corresponding norm $\|u\|_V$. Let ω be a subdomain of Ω. We define $(u, v)_{V(\omega)}$ to be the inner product for functions obtained by restricting on ω.

Suppose $\mathcal{B}(\cdot, \cdot)$ is a bilinear form on $V \times V$ and $\mathcal{F}(\cdot)$ a linear functional on V such that

(i) $\mathcal{B}(\cdot, \cdot)$ is continuous, i.e. there exists a constant $C > 0$ such that

$$| \mathcal{B}(u, v) | \leq C \|u\|_{V(\omega)} \|v\|_{V(\omega)}, \quad \forall u, v \in V,$$

where $\omega = \{\text{supp } u\} \cap \{\text{supp } v\}$.

(ii) $\mathcal{B}(\cdot,\cdot)$ is V-elliptic, i.e. there exists a constant $c > 0$ such that

$$\mathcal{B}(u,u) \geq c\|u\|_V^2, \quad \forall u \in V.$$

(iii) $\mathcal{F}(\cdot)$ is continuous, i.e. there exists a constant $C > 0$ such that

$$| \mathcal{F}(u) | \leq C\|u\|_V, \quad \forall u \in V.$$

We define

$$\mathcal{A}(u,v) = 1/2(\mathcal{B}(u,v) + \mathcal{B}(v,u)),$$

which will be called the symmetric part of $\mathcal{B}(\cdot,\cdot)$, and

$$\mathcal{N}(u,v) = 1/2(\mathcal{B}(u,v) - \mathcal{B}(v,u)),$$

which will be called the skewsymmetric part of $\mathcal{B}(\cdot,\cdot)$.

We assume, following from the assumptions on $\mathcal{B}(u,v)$, that $\mathcal{A}(u,v)$ is continuous and elliptic in the same sense as in (i) and (ii), which implies that the norm corresponding to $\mathcal{A}(\cdot,\cdot)$ is equivalent to the V-norm. In the following, we shall use $(\cdot,\cdot)_{\mathcal{A}}$ instead of $(\cdot,\cdot)_V$.

Our abstract variational equation reads as follows: Find $\mathbf{u} \in V$, such that

$$\mathcal{B}(\mathbf{u},v) \;=\; \mathcal{F}(v), \quad \forall v \in V. \tag{1.3}$$

In order to define the additive Schwarz method, we assume that there exists a decomposition of V. Let V_i, $i = 0, \cdots, N$, be subspaces of V, such that

$$V \;=\; \sum_{i=0}^{N} V_i.$$

Moreover, we assume that there exists a constant $C_0 > 0$ such that for any $v \in V$, there exist $v_i \in V_i$, $i = 0, \cdots, N$, such that

$$v = \sum_{i=0}^{N} v_i \tag{1.4}$$

and

$$\sum_{i=0}^{N} \|v_i\|^2_{V(\omega_i)} \leq C_0^2 \|v\|^2_V, \quad \forall v \in V, \tag{1.5}$$

where ω_i is the support of V_i defined as

$$\omega_i = \{x | \in \Omega, \exists u \in V_i \text{ such that } u(x) \neq 0\}.$$

Note that the constant C_0 may depend on the number of subregions N and also some other parameters of V, which may be introduced in practical applications. By using this function space decomposition $\{V_i\}$ we split the original problem defined on V, which usually has a large number of degree of freedoms, into some smaller problems defined on V_i, which are chosen to have relatively small number of degree of freedoms. These small problems are usually independent of each other therefore can be solved in parallel.

We also assume that the supports of $\{\omega_i\}$ form a finite covering of Ω in the sense that there exists a constant $C_\omega > 0$, such that

$$\sum_{i=0}^{N} \|v\|^2_{V(\omega_i)} \leq C_\omega \|v\|^2_V, \quad \forall v \in V. \tag{1.6}$$

In fact this constant C_ω is the maximum number of ω_i's to which a point $x \in \Omega$ can belong.

For each subspace V_i, $0 \leq i \leq N$, we define a projection

$$P_i^{\mathcal{B}} = P_{V_i}^{\mathcal{B}} : \ V \longrightarrow V_i$$

with respect to the bilinear form $\mathcal{B}(\cdot, \cdot)$: For any given $u \in V$, $P_i^{\mathcal{B}} u \in V_i$ is the solution of the problem

$$\mathcal{B}(P_i^{\mathcal{B}} u, v) = \mathcal{B}(u, v), \ \forall v \in V_i.$$

We introduce a mapping $P^{\mathcal{B}} : V \longrightarrow V$ as

$$P^{\mathcal{B}} = P_0^{\mathcal{B}} + \cdots + P_N^{\mathcal{B}},$$

which is the main operator that we shall study in this thesis.

Let $\mathbf{u}$ be the solution of (1.3), and denote the image of $\mathbf{u}$ under the mapping $P^{\mathcal{B}}$ as

$$\mathbf{b} = \sum_{i=0}^{N} P_i^{\mathcal{B}} \mathbf{u}.$$

It is easy to see that $\mathbf{b}$ can be computed without knowing the solution $\mathbf{u}$ itself. We compute $P_i^{\mathcal{B}}\mathbf{u}$ by solving the linear system of equations

$$\mathcal{B}(P_i^{\mathcal{B}}\mathbf{u}, v) = \mathcal{F}(v), \quad \forall v \in V_i,$$

for each i. $\mathbf{b}$ is thus obtained by taking the sum of the solutions.

We now introduce a new linear system of equations: Find $\mathbf{u} \in V$, such that

$$P^{\mathcal{B}}\mathbf{u} = \mathbf{b}. \tag{1.7}$$

We shall call this equation the derived equation with respect to the bilinear form $\mathcal{B}(\cdot,\cdot)$ and the decomposition $\{V_i\}$.

The following theorem can be established trivially.

Theorem 1.2 *If $P^{\mathcal{B}}$ is invertiable, then the equations (1.3) and (1.7) have the same solution.*

The additive Schwarz algorithm can be stated as

Additive Schwarz Algorithm: Find the solution $\mathbf{u}$ of equation (1.3) by solving equation (1.7).

In practice, the operator $P^{\mathcal{B}}$ usually corresponds to the sum of products of a large, sparse matrix and the inverses of some other sparse matrices. The explicit matrix for $P^{\mathcal{B}}$ is not known except in some special cases, however, the matrix vector multiply $P^{\mathcal{B}}u$ can be computed by solving one linear system of equations for each subregion. Therefore iterative methods

are natural candidates for problem (1.7). The rate of convergence of any iterative methods depends on the conditioning of the operator $P^{\mathcal{B}}$.

Theorem 1.3 *(1) There exists a constant $C > 0$, such that*

$$\|P^{\mathcal{B}}u\|_{\mathcal{A}} \leq CC_{\omega}\|u\|_{\mathcal{A}}, \quad \forall u \in V.$$

(2) There exists a constant $c_1 > 0$, independent of C_0, such that

$$\|P^{\mathcal{B}}u\|_{\mathcal{A}} \geq c_1 C_0^{-2}\|u\|_{\mathcal{A}}, \quad \forall u \in V.$$

(3) If there exists $0 < \delta < 1$, such that $|\mathcal{N}(u, P^{\mathcal{B}}u)| \leq \delta\mathcal{B}(u, P^{\mathcal{B}}u)$, $\forall u \in V$, then

$$(u, P^{\mathcal{B}}u)_{\mathcal{A}} \geq c_2(1-\delta)C_0^{-2}(u,u)_{\mathcal{A}}, \quad \forall u \in V,$$

where $c_2 > 0$ is independent of C_0 and δ.

Proof:

(1) We first show that there exists a constant $C > 0$, such that

$$\sum_{i=0}^{N} \| P_i^{\mathcal{B}}u \|^2_{\mathcal{A}(\omega_i)} \leq CC_{\omega}\| u \|^2_{\mathcal{A}}, \quad \forall u \in V. \tag{1.8}$$

By the definition of the $\mathcal{A}$-norm,

$$\sum_{i=0}^{N} \| P_i^{\mathcal{B}}u \|^2_{\mathcal{A}(\omega_i)} = \sum_{i=0}^{N} \mathcal{B}(P_i^{\mathcal{B}}u, P_i^{\mathcal{B}}u), \quad \forall u \in V.$$

By the definition of the projection $P_i^{\mathcal{B}}$, we have

$$\mathcal{B}(P_i^{\mathcal{B}}u, P_i^{\mathcal{B}}u) = \mathcal{B}(u, P_i^{\mathcal{B}}u).$$

Since $P_i^{\mathcal{B}}u$ is zero outside ω_i, we have, by the boundedness assumption (i), that

$$\mathcal{B}(u, P_i^{\mathcal{B}} u) \;\leq\; C \| u \|_{\mathcal{A}(\omega_i)} \cdot \| P_i^{\mathcal{B}} u \|_{\mathcal{A}(\omega_i)}.$$

Combining the above results and applying Cauchy's inequality, we obtain

$$\sum_{i=0}^{N} \| P_i^{\mathcal{B}} u \|^2_{\mathcal{A}(\omega_i)} \;\leq\; C \sqrt{\sum_{i=0}^{N} \| u \|^2_{\mathcal{A}(\omega_i)}} \cdot \sqrt{\sum_{i=0}^{N} \| P_i^{\mathcal{B}} u \|^2_{\mathcal{A}(\omega_i)}}.$$

If we cancel the common factor on both sides, and use the assumption (1.6), we obtain the estimate (1.8).

It now suffices to bound $\| P^{\mathcal{B}} u \|_{\mathcal{A}}$ by the left hand side of estimate (1.8). Using the definition of $P^{\mathcal{B}}$, we have

$$\| P^{\mathcal{B}} u \|^2_{\mathcal{A}} \;=\; \mathcal{B}(P^{\mathcal{B}} u, P^{\mathcal{B}} u) \;=\; \sum_{i=0}^{N} \mathcal{B}(P^{\mathcal{B}} u, P_i^{\mathcal{B}} u).$$

Note that the function $P_i^{\mathcal{B}} u$ has its support only in ω_i. Thus

$$\mathcal{B}(P^{\mathcal{B}} u, P_i^{\mathcal{B}} u) \;\leq\; C \| P^{\mathcal{B}} u \|_{\mathcal{A}(\omega_i)} \cdot \| P_i^{\mathcal{B}} u \|_{\mathcal{A}(\omega_i)}.$$

Applying Cauchy's inequality and the assumption (1.6), we obtain

$$\| P^{\mathcal{B}} u \|^2_{\mathcal{A}} \;\leq\; C C_\omega \sum_{i=0}^{N} \| P_i^{\mathcal{B}} u \|^2_{\mathcal{A}(\omega_i)}.$$

Combined with (1.8), we can conclude that

$$\| P^{\mathcal{B}} u \|_{\mathcal{A}} \;\leq\; C C_\omega \| u \|_{\mathcal{A}},$$

which proves the upper bound part of the theorem.

(2) We now turn to the proof of the lower bound part of the Theorem 1.3. By the V-function decomposition assumption (1.4), any $u \in V$ can be written as

$$u = \sum_{i=0}^{N} u_i,$$

where $u_i \in V_i$. This decomposition is chosen so that it satisfies (1.5). We have

$$\| u \|_{\mathcal{A}}^2 = \mathcal{B}(u,u) = \sum_{i=0}^{N} \mathcal{B}(u,u_i).$$

Using the definition of the projection $P_i^{\mathcal{B}}$, we obtain

$$\| u \|_{\mathcal{A}}^2 = \sum_{i=1}^{N} \mathcal{B}(P_i^{\mathcal{B}} u, u_i)$$

$$\leq C \sum_{i=0}^{N} \| P_i^{\mathcal{B}} u \|_{\mathcal{A}(\omega_i)} \cdot \| u_i \|_{\mathcal{A}(\omega_i)}.$$

Applying Cauchy's inequality and the bounded decomposition assumption (1.5), we obtain

$$\| u \|_{\mathcal{A}}^2 \leq C \cdot C_0^2 \sum_{i=0}^{N} \| P_i^{\mathcal{B}} u \|_{\mathcal{A}(\omega_i)}^2, \tag{1.9}$$

where $C > 0$ is independent of C_0. By the definition of the $\mathcal{A}$-norm,

$$\sum_{i=0}^{N} \| P_i^{\mathcal{B}} u \|_{\mathcal{A}(\omega_i)}^2 = \sum_{i=0}^{N} \mathcal{B}(P_i^{\mathcal{B}} u, P_i^{\mathcal{B}} u)$$

$$= \sum_{i=0}^{N} \mathcal{B}(u, P_i^{\mathcal{B}} u).$$

Because of the linearity of $\mathcal{B}(\cdot,\cdot)$, we have

$$\sum_{i=0}^{N} \| P_i^{\mathcal{B}} u \|_{\mathcal{A}(\omega_i)}^2 = \mathcal{B}(u, P^{\mathcal{B}} u) \leq C \| u \|_{\mathcal{A}} \| P^{\mathcal{B}} u \|_{\mathcal{A}}. \tag{1.10}$$

Combined with the estimate (1.9), we have

$$\| u \|_{\mathcal{A}} \leq C \cdot C_0^2 \| P^{\mathcal{B}} u \|_{\mathcal{A}},$$

where C is independent of C_0.

(3) It is easy to verify that

$$(u, P^{\mathcal{B}} u)_{\mathcal{A}} = \mathcal{B}(u, P^{\mathcal{B}} u) - \mathcal{N}(u, P^{\mathcal{B}} u).$$

By the assumption on the bilinear form $\mathcal{N}(\cdot, \cdot)$, the right hand side can be bounded from below by

$$(1 - \delta)\mathcal{B}(u, P^{\mathcal{B}} u).$$

Since $P^{\mathcal{B}}$ is the sum of the $P_i^{\mathcal{B}}$'s, the above expression equals

$$(1 - \delta) \sum_{i=0}^{N} \mathcal{B}(P_i^{\mathcal{B}} u, P_i^{\mathcal{B}} u).$$

By assumption (ii), it can be bounded from below by

$$c(1 - \delta) \sum_{i=0}^{N} \| P_i^{\mathcal{B}} u \|^2_{\mathcal{A}(\omega_i)}.$$

The constant $c_2(C_0) = C(1 - \delta)C_0^{-2} > 0$ can be obtained by using the estimate (1.9) and $C > 0$ is independent of C_0. □

Chapter 2

Methods for stationary convection-diffusion problems

In this chapter, we study two Schwarz type methods for the stationary convection-diffusion equations; the additive Schwarz method ($AS\!M$) and the iterative substructuring method ($IS\!M$). Both methods fall in the framework provided by the abstract additive Schwarz theory, however, the analysis and computational implementation are different. We prove that $AS\!M$is an optimal algorithm in the sense that the rate of convergence is independent of the mesh parameters provided that the size of the substructures satisfies some conditions. This analysis is valid in both R^2 and R^3. We also give a convergence rate analysis for the $IS\!M$for the R^2 case. The bound depends only mildly on the mesh parameters.

2.1 Introduction

2.1.1 A stationary convection-diffusion problem

In this section, we present a continuous second order linear elliptic partial differential equation, which is not necessarily selfadjoint, in a bounded

region in R^2 or R^3. We also give the variational formulation, basic assumptions and some classical regularity results, which are important for the development of our theory.

Let Ω be an open bounded polygon in R^2 or R^3, with boundary $\partial\Omega$. d denotes the dimension of the space, $d = 2$ or $d = 3$. Consider the homogenous Dirichlet boundary value problem

$$\begin{cases} Lu &= f \quad \text{in} \quad \Omega \\ u &= 0 \quad \text{on} \quad \partial\Omega, \end{cases} \tag{2.1}$$

where L is a strongly elliptic operator of the following form

$$Lu(x) = -\sum_{i,j=1}^{d} \frac{\partial}{\partial x_i}(a_{ij}(x)\frac{\partial u(x)}{\partial x_j}) + \sum_{i=1}^{d} b_i(x)\frac{\partial u(x)}{\partial x_i} + c(x)u(x),$$

where $a_{ij}(x) = a_{ji}(x)$ for all i, j and $x \in \Omega$.

Since the adjoint problem of (2.1) will be used in later chapters, we shall write down an integral formula. The adjoint operator, denoted as L^*, satisfies the relation:

$$(Lu, v) = (u, L^*v), \quad \forall u, v \in H_0^1(\Omega).$$

By applying Green's identity, we can easily find the formula for L^*,

$$L^*v = -\sum_{i,j=1}^{d} \frac{\partial}{\partial x_i}(a_{ij}(x)\frac{\partial v}{\partial x_j}) - \sum_{i=1}^{d} \frac{\partial}{\partial x_i}(b_i(x)v) + c(x)v.$$

We assume that $f \in L^2(\Omega)$. The existence and uniqueness of the solution of equation (2.1), as well as its adjoint equation, are well understood. It is well known that if the domain boundary is not smooth enough, for example on a polygonal domain and the domain is not convex, we cannot expect, in general, the solution to be in $H^2(\Omega)$. According to the classical elliptic regularity theory on Lipschitz region, see [1] we make our regularity

assumption as follows: There exists a constant $\gamma \in (0, 1/2)$, which depends on the geometry of Ω, such that the adjoint equation has a unique solution $u \in H^{1+\gamma}(\Omega) \cap H_0^1(\Omega)$, and furthermore there exists a constant C, such that

$$\|u\|_{H^{1+\gamma}(\Omega)} \leq C\|L^* u\|_{L^2(\Omega)}. \tag{2.2}$$

The weak form of equation (2.1) reads as follows: Find u such that

$$B(u, v) = F(v), \qquad \forall v \in H_0^1(\Omega), \tag{2.3}$$

where the bilinear form is defined by

$$B(u, v) = \sum_{i,j=1}^{d} \int_\Omega a_{ij} \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j} dx + \sum_{i=1}^{d} \int_\Omega b_i \frac{\partial u}{\partial x_i} v dx + \int_\Omega cuv dx$$

and the linear functional F is defined by

$$F(v) = \int_\Omega fv dx \quad \text{with} \quad v \in H_0^1(\Omega).$$

We assume that this bilinear form is bounded and strongly elliptic, i.e. there exist two constants $C > 0$ and $c > 0$, such that

$$| B(u, v) | \leq C\| u \|_{H_0^1(\Omega)} \| v \|_{H_0^1(\Omega)}, \quad \forall u, v \in H_0^1(\Omega), \tag{2.4}$$

and

$$B(u, u) \geq c\| u \|^2_{H_0^1(\Omega)}, \quad \forall u \in H_0^1(\Omega). \tag{2.5}$$

If the coefficients of the first order terms in equation (2.1) are identically zero, then the differential operator L is selfadjoint. To separate the symmetric part and the skewsymmetric part of the bilinear form $B(\cdot, \cdot)$, we introduce two bilinear forms

(1) The symmetric part

$$A(u,v) = \frac{1}{2}(B(u,v) + B(v,u))$$

$$= \sum_{i,j=1}^{d} \int_\Omega a_{ij} \partial u/\partial x_i \partial v/\partial x_j dx + \int_\Omega c - \frac{1}{2} div(b) uv dx$$

(2) The skewsymmetric part

$$N(u,v) = \frac{1}{2}(B(u,v) - B(v,u))$$

$$= \sum_{i=1}^{d} \int_\Omega b_i \partial u/\partial x_i v dx + \frac{1}{2} \int_\Omega div(b) uv dx.$$

From the boundness assumption on $B(\cdot,\cdot)$ and the Schwarz inequality, it follows that there exists a constant $C > 0$ such that

$$| A(u,v) | \leq C \| u \|_{H_0^1(\Omega)} \| v \|_{H_0^1(\Omega)}, \quad \forall u, v \in H_0^1(\Omega),$$

and from the ellipticity assumption, there exists another constant $c > 0$ such that

$$A(u,u) \geq c \| u \|^2_{H_0^1(\Omega)}, \quad \forall u \in H_0^1(\Omega).$$

Remark: The boundness and ellipticity of $A(\cdot,\cdot)$ imply that the A-norm, defined by $\sqrt{A(\cdot,\cdot)}$, is equivalent to the $H_0^1(\Omega)$ norm.

We assume that the bilinear form $N(\cdot,\cdot)$ is bounded, i.e. there exists a constant C, such that

$$| N(u,v) | \leq C \| u \|_{H_0^1(\Omega)} \| v \|_{L^2(\Omega)}, \qquad \forall u, v \in H_0^1(\Omega).$$

This bound can be established if the coefficients $b_i(x)$ and $div(b)$ are bounded. We note that the estimates for $A(\cdot,\cdot)$ and $N(\cdot,\cdot)$ are different. If we look at the integral formulas for $A(\cdot,\cdot)$ and $N(\cdot,\cdot)$, it is clear that the terms in $N(\cdot,\cdot)$ are one order lower than the terms in $A(\cdot,\cdot)$. This is a key factor that makes our proofs work.

2.1.2 The Galerkin finite element method

We solve equation (2.3) by a conformal Galerkin finite element method. For simplicity, we use piecewise linear triangular element in R^2 and the corresponding tetrahedronal element in R^3. In this subsection, we first introduce a two level triangulation of Ω and the corresponding finite element spaces. We also discuss some basic properties of the finite element spaces.

1. A Two level triangulation

For a given polygonal region $\Omega \in R^d$, in the first step, we define $\{\Omega_i\}$ to be a regular finite element triangulation of Ω where $\{\Omega_i\}$ is a set of non-overlapping d-dimensional simplices, i.e. triangles if $d = 2$ and tetrahedra if $d = 3$, such that no vertex of one simplex lies on an edge or a face of another and

$$\overline{\Omega} = \bigcup_{i=1}^{N} \overline{\Omega_i}.$$

Here N is the number of simplices.

Let H_i denote the diameter of Ω_i and $\tilde{H}_i$ denote the diameter of the largest inscribed ball in Ω_i. We assume that the ratio $H_i/\tilde{H}_i$ is uniformly bounded from above, i.e. the discretization is shape regular.

We introduce the mesh parameter

$$H = \max\{H_1, \cdots, H_N\}.$$

We call Ω_i a substructure and $\{\Omega_i\}$ the coarse mesh or H-level subdivision of Ω.

In our second step, we further divide each substructure Ω_i into smaller simplices, denoted as τ_i^j, $j = 1, \cdots$. We assume that $\{\tau_i^j\}$ form a shape regular finite element triangulation in the same sense as above. Let h_i^j be

the diameter of τ_i^j, we introduce the fine mesh parameter

$$h = \max_{i,j}(h_i^j).$$

We call $\cup_{i,j}\tau_i^j$ the fine mesh or h-level subdivision of Ω .

Next, we define the piecewise linear finite element function spaces over both H-level and h-level subdivision of Ω.

$$V^H = \{v^H \mid \text{ continuous on } \Omega,\ v^H \mid_{\Omega_i} \text{ linear on } \Omega_i,\ v^H = 0 \text{ on } \partial\Omega\}$$

$$V^h = \{v^h \mid \text{ continuous on } \Omega,\ v^h \mid_{\tau_i^j} \text{ linear on } \tau_i^j,\ v^h = 0 \text{ on } \partial\Omega\}$$

It is obvious that $V^H \subset V^h$. To simplify the notations, we denote

$$\Lambda^h = \{x \mid \in \text{ interior nodes of } h - \text{level subdivision}\}$$

$$\Lambda^H = \{x \mid \in \text{ interior nodes of } H - \text{level subdivision}\}.$$

Associated with each node $x_i \in \Lambda^h$, there is a nodal base function, denoted as ϕ_i^h, such that $\phi_i^h(x) \in V^h$ and $\phi_i^h(x_j) = \delta_{ij}$. The set $\{\phi_i^h\}$ forms the nodal base of V^h.

2. Galerkin finite element approximation

The Galerkin approximation of equation (2.3) reads as follows: Find $\mathbf{u}^h \in V^h$, such that

$$B(\mathbf{u}^h, v^h) = F(v^h),\ \ \forall v^h \in V^h. \tag{2.6}$$

The existence and uniqueness of $\mathbf{u}^h$ has been extensively studied in the literature, see [1]. By using the nodal base functions, equation (2.6) can be transformed into a linear system of equations, which is usually large and sparse. It is well known that the efficiency of any iterative methods to be used to solve this linear system of equations depends strongly on the conditioning of the stiffness matrix K_h, where $K_h = (B(\phi_i^h, \phi_j^h))$.

2.2 Additive Schwarz method for the problems in R^2 and R^3

2.2.1 An algorithm and the main results

The additive variant of the Schwarz alternating method was originally proposed by M. Dryja and O. Widlund [11] and S. Nepomnyaschikh [26] for selfadjoint stationary elliptic problems. In this section, we generalize this method to the nonselfadjoint elliptic case.

We first form a basic decomposition of the domain Ω and we then define the projections which will lead to our additive Schwarz algorithm.

1. Basic domain decomposition and projections

In the previous section, we introduced the H-level subdivision $\{\Omega_i\}$ of Ω. Since the Schwarz type domain decomposition methods use overlapping subregions, we extend each subregion Ω_i to a larger region Ω_i' such that

$$\Omega_i \subset \Omega_i'.$$

Moreover, we assume that there exists a constant $\alpha > 0$ such that

$$\text{distance}(\partial\Omega_i' \cap \Omega, \partial\Omega_i \cap Omega) \geq \alpha H_i, \quad \forall i.$$

To simplify the notation, we denote $\Omega_0' = \Omega$.

We suppose that $\partial\Omega_i'$ does not cut through any h-level elements. We make the same constructions for the subregions that meet the boundary except that we cut off the parts that are outside Ω.

The assumption on the existence of such a constant α is important. From the estimates, which will be given later in this chapter, we can see that the larger the α the better our lower bound on the spectrum will be. But if we increase the overlap, the size of the subproblems also increases,

therefore, the cost for solving the subproblems in each iteration will be increased. To balance the total number of iteration and the cost of solving the subproblems is an important practical issue.

For each Ω_i', there is a regular finite element subdivision which is naturally induced from the h-level subdivision of Ω. The corresponding finite element function space, denoted by V_i^h, is defined as

$$V_i^h = V^h \cap H_0^1(\Omega_i'),$$

which can be regarded as a subspace of V^h if we extend each function by zero to the complement of Ω_i'. It is known that this extension is continuous. In the case when $\partial\Omega_i'$ intersects $\partial\Omega$, we use the original boundary condition on $\partial\Omega_i' \bigcap \Omega$.

To make our notations simple, we let

$$V_0^h = V^H.$$

When convenient, we can regard any element of V_0^h as a coarse mesh interpolation $I_H u^h$ of some elements $u^h \in V^h$, which uses the function values at the nodes of the coarse mesh only.

It can be easily seen that our finite element function space V^h can be represented as the sum of the $N+1$ subspaces, i.e.

$$V^h = V_0^h + V_1^h + \cdots + V_N^h.$$

Let $P_{V_i^h}^B = P_i^B$ denote the projection from the finite element space V^h to the subspace V_i^h with respect to the bilinear form $B(\cdot,\cdot)$, and define the mapping $P^B : V^h \to V^h$ as

$$P^B = P_0^B + P_1^B + \cdots + P_N^B.$$

This is the main operator in our further studies in the chapter.

It is easy to see that the computation of the projection of an arbitrary function $v^h \in V^h$ into the subspace V_i^h involves only the solution of a standard finite element linear system of algebraic equations on Ω_i', which is a small subregion in the case $i \neq 0$. If $i = 0$, it is the standard finite element equation on the H-level coarse space.

Let us denote

$$\mathbf{b} = P^B \mathbf{u}^h = \sum_{i=0}^{N} P_i^B \mathbf{u}^h.$$

By taking $\mathbf{b}$ as the right hand side, we can define our derived linear system with respect to the bilinear form $B(\cdot,\cdot)$ and the decomposition $\{V_i\}$ as follows,

$$P^B \mathbf{u}^h = \mathbf{b}. \tag{2.7}$$

By Theorem 1.2, the equation (2.7) is equivalent to the Galerkin equation (2.6). If the operator L is not selfadjoint, then P^B is not symmetric.

2. An algorithm and the main theorem

Due to the special properties of P^B, any numerical algorithms designed to solve such a linear system of equations must be robust with respect to asymmetry and require matrix vector multiplications only. The GMRES method described in Chapter 1 is one such algorithm which we shall use in this paper. Chebyshev iterative method (see [23]) would provide an alternative, which will not be considered here.

Additive Schwarz algorithm: Solving equation (2.7) by GMRES method with the inner product $[\cdot,\cdot] = (\cdot,\cdot)_A$.

According to Theorem 1.1, the rate of convergence of this algorithm can

be estimated by certain spectral bounds for the operator P^B. One of the main theorems of this chapter shows that the operator P^B of the equation (2.7) is uniformly well conditioned in the sense that all the bounds are independent of the mesh parameters H and h. The proof is given in the next section.

We adopt the following convention. All the constants, denoted as C, c, c_p, C_P etc., are positive and independent of the mesh parameters H and h.

Theorem 2.1 *The operator P^B is uniformly well conditioned in the following sense:*

(1) There exists a constant C_p, such that

$$\| P^B u^h \|_A \leq C_p \| u^h \|_A, \quad \forall u^h \in V^h.$$

(2) There exists a constant c, such that

$$\| P^B u^h \|_A \geq c \| u^h \|_A, \quad \forall u^h \in V^h.$$

(3) There exists a constant $H_0 > 0$, independent of H and h, and a constant $c_p(H_0)$, such that for $H \leq H_0$,

$$(u^h, P^B u^h)_A \geq c_p (u^h, u^h)_A, \quad \forall u^h \in V^h.$$

Remarks: (a) The operator P_0^B is very important. It provides for global information transportation. All the other P_i^Bs are local mappings. Without using P_0^B, in each iteration the information travels only from one substructure to its next neighbors. Therefore, it takes at least $O(1/H)$ iterations for the information to move across the region.

(b) In the case that $B(\cdot, \cdot)$ is symmetric, this additive Schwarz algorithm is identical to the one proposed by M. Dryja and O. Widlund. Theorem 2.1, part (1) and (2), gives the convergence rate estimates.

(c) Theorem 2.1 part (3) shows that if the coarse mesh size is fine enough, then the symmetric part is positive definite, which guarantees that the GMRES method converges. Since both constants c_p and C_p are independent of the mesh parameters H and h, then according to Theorem 1.1, the convergence rate of the GMRES method will not dependent on the size of the discrete problems.

(d) The constant H_0 determines the size of the coarse mesh problem. H_0 depends on the problem. In general, H_0 decreases if we increase the coefficients of the nonselfadjoint terms, while it increases if we use larger overlap. It also depends on the shape of the domain Ω. We do not have an explicit formula for H_0. However, we will give some idea about how to determine this constant numerically later in the chapter of numerical results.

(e) If the skewsymmetric part of the elliptic operator vanishes, then

$$H_0 = +\infty,$$

and hence we have no restrictions on the coarse mesh size H.

2.2.2 The condition number estimates

In order to prove Theorem 2.1, we need only to show that all the assumptions for Theorem 1.3 hold. Since $\{\Omega_i'\}$ is a finite covering of Ω, the maximum number of Ω_i's to which any point in Ω can belong is a finite number, denoted as C_ω. It is easy to verify that

$$\sum_{i=0}^{N} \| u^h \|^2_{A(\Omega_i')} \leq C_\omega \| u^h \|^2_A, \quad \forall u^h \in V^h, \tag{2.8}$$

i.e. assumption (1.6) holds.

Note that if we increase the size of the overlap between the basic subregion Ω_i's, then this constant C_ω increases.

Next, we show that the assumption (1.5) holds. Let us introduce the coarse mesh L^2 projection as follows: For any $u^h \in V^h$, the L^2 projection $\hat{I}_H u^h \in V^H$ is the solution of the following linear system

$$(\hat{I}_H u^h, v^H) \;=\; (u^h, v^H), \;\; \forall v^H \in V^H.$$

Some useful properties about the L^2 projection are summarized in the following lemma, which holds in both R^2 and R^3.

Lemma 2.1 *There exist two constants $C_1 > 0$ and $C_2 > 0$, independent of H and h, such that*

$$\|\hat{I}_H u^h\|_A \leq C_1 \|u^h\|_A, \;\; \forall u^h \in V^h,$$

$$\|u^h - \hat{I}_H u^h\|_{L^2(\Omega)} \leq C_2 H \|u^h\|_A, \;\; \forall u^h \in V^h.$$

Proof: See [27]. □

Lemma 2.2 *For any $u^h \in V^h$, there exist $u_i^h \in V_i^h(\Omega_i')$, such that*

$$u^h \;=\; \sum_{i=0}^{N} u_i^h.$$

Moreover, there exists a constant C_0, such that

$$\sum_{i=0}^{N} \| u_i^h \|^2_{A(\Omega_i')} \;\leq\; C_0^2 \| u^h \|^2_A.$$

Proof: Following M. Dryja and O. Widlund, we first construct the decomposition and then prove that it satisfies the estimate.

For a given $u^h \in V^h$, we take u_0^h to be the L^2 projection

$$u_0^h \;=\; \hat{I}_H u^h.$$

Let us denote

$$w^h \equiv u^h - \hat{I}_H u^h.$$

Following the paper of M. Dryja and O. Widlund [11], the functions u_i^h mentioned in Lemma 2.2 can be constructed easily. We take

$$u_i^h = I_h(\theta_i w_h), \quad i = 1, \cdots, N,$$

where $\{\theta_i\}$ defines a partition of unity of Ω and belongs to $C_0^\infty(\Omega_i')$ and I_h is the interpolation operator at the h-level nodal points. We can arrange so that $\nabla\theta_i$ is bounded by $const/H_i$. By using the linearity of I_h, we can easily show that we have obtained a correct decomposition of u^h. According to the theory of M. Dryja and O. Widlund [11], we have

$$| u_i^h |^2_{A(\Omega_i')} \leq C(| w_h |^2_{A(\Omega_i')} + H^{-2} \| w_h \|^2_{L^2(\Omega_i')}),$$

which holds for both $2D$ and $3D$. We can add up them for $i = 1, \cdots, N$ and obtain an estimate on Ω,

$$\sum_{i=1}^{N} | u_i^h |^2_{A(\Omega_i')} \leq C(| w_h |^2_A + \frac{1}{H^2} \| w_h \|^2_{L^2(\Omega)}).$$

According to Lemma 2.1, we have

$$\| w^h \|^2_{L^2(\Omega)} \leq CH^2 | u^h |^2_A.$$

The proof follows by combining these estimates.□

In order to prove part (3), we need the following estimates for the skewsymmetric part.

Lemma 2.3 *There exists a constant C, such that*

(a) $| N(P_0^B u^h, u^h - P_0^B u^h) | \leq CH^\gamma \| P_0^B u^h \|_A \| u^h \|_A, \ \forall u^h \in V^h.$

$$(b) \quad | N(P_i^B u^h, u^h - P_i^B u^h) | \leq CH \parallel P_i^B u^h \parallel_{A(\Omega_i')} \parallel u^h \parallel_{A(\Omega_i')}, \quad \forall u^h \in V^h$$
$$for\ i = 1, \cdots, N.$$

Proof: (a) By the boundness assumption on $N(\cdot, \cdot)$, we need only to show that

$$\parallel u^h - P_0^B u^h \parallel_{L^2(\Omega)} \leq CH^\gamma \parallel u^h \parallel_A,$$

where $C > 0$ does not dependent on H and h. From the identity

$$B(P_0^B u^h - u^h, P_0^B u^h - u^h) = -B(P_0^B u^h - u^h, u^h),$$

we can easily obtain the A-norm estimate, i.e. there exists a constant $C > 0$ such that

$$\parallel u^h - P_0^B u^h \parallel_A \leq C \parallel u^h \parallel_A .$$

The L^2 estimate is obtained by using the so called Nitsche trick. We have

$$\parallel P_0^B u^h - u^h \parallel_{L^2(\Omega)} = \sup_{\|v\|_{L^2} \neq 0} \frac{(P_0^B u^h - u^h, v)}{\parallel v \parallel_{L^2(\Omega)}}. \tag{2.9}$$

For any fixed $v \in L^2(\Omega)$, we introduce the following auxiliary variational problem: Find w, such that

$$B(\phi, w) = (v, \phi), \quad \forall \phi \in H_0^1(\Omega).$$

By our assumptions the solution exists and satisfies

$$\parallel w \parallel_{H^{1+\gamma}(\Omega)} \leq C \parallel v \parallel_{L^2(\Omega)} .$$

If we take the test function ϕ in the above equation to be $P_0^B u^h - u^h$ and also use the definition of the projection P_0^B, we have

$$B(P_0^B u^h - u^h, w - I_H w) = (v, P_0^B u^h - u^h),$$

where I_H is the usual interpolation operator at the H-level nodal points in Ω. By the well known result in finite element approximation theory, we have that

$$\| w - I_H w \|_{H_0^1(\Omega)} \leq C H^\gamma \mid w \mid_{H^{1+\gamma}(\Omega)} \leq C H^\gamma \| v \|_{L^2(\Omega)} .$$

Therefore, we have

$$\mid B(P_0^B u^h - u^h, w - I_H w) \mid \leq C H^\gamma \| v \|_{L^2(\Omega)} \| P_0^B u^h - u^h \|_{H_0^1(\Omega)} .$$

By substituting this estimate into (2.9), we obtain the desired proof of (a).

(b) By the boundedness assumption on the bilinear form $N(\cdot,\cdot)$, we have

$$\mid N(P_i^B u^h, u^h - P_i^B u^h) \mid \leq C \| P_i^B u^h \|_{L^2(\Omega_i')} (\| u^h \|_{A(\Omega_i')} + \| P_i^B u^h \|_{A(\Omega_i')}).$$

It is easy to see that

$$\| P_i^B u^h \|^2_{A(\Omega_i')} = B(P_i^B u^h, P_i^B u^h)$$

$$= B(u^h, P_i^B u^h) \leq C \| u^h \|_{A(\Omega_i')} \| P_i^B u^h \|_{A(\Omega_i')},$$

which implies that

$$\| P_i^B u^h \|_{A(\Omega_i')} \leq C \| u^h \|_{A(\Omega_i')} .$$

The proof of (b) follows from combining the above estimates and Poincaré's inequality. □

Remark: It is easy to verify that

$$N(u^h, P u^h) = \sum_{i=0}^{N} N(u, P_i u^h) = -\sum_{i=0}^{N} N(P_i u^h, u^h - P_i u^h).$$

Using Lemma 2.3, we can estimate the right hand side of the above expression by

$$CH^{\gamma}\sum_{i=0}^{N} \| P_i u^h \|_{A(\Omega_i')} \| u^h \|_{A(\Omega_i')} .$$

Applying Cauchy's inequality and estimate (1.9), we have that the above expression is bounded by

$$CH^{\gamma}(\sum_{i=0}^{N} \| P_i u^h \|^2_{A(\Omega_i')}) = CH^{\gamma}\sum_{i=0}^{N} B(P_i u^h, P_i u^h)$$

$$= CH^{\gamma} B(u^h, Pu^h)$$

We take $H_0 = (1/C)^{1/\gamma}$. Thus if $H < H_0$, then $\delta \equiv CH^{\gamma} < 1$, therefore, the assumption for Theorem 1.3 part (3) holds.

The independency of the mesh parameters can be accomplished by examining all the constants appeared in the above discussion. The proof for Theorem 2.1 is thus completed.

2.3 Iterative substructuring method for the problems in R^2

2.3.1 An algorithm and the main results

1. Basic domain decomposition and the projections

Recall that in the previous section, we extended each H-level substructure Ω_i to a large region Ω_i', on which we define our projection and the corresponding subproblem. $\{\Omega_i'\}$ is the basic decomposition of the domain. Now, instead of extending each region, we combine each pair of adjacent substructures to form our basic decomposition of Ω. We need to introduce some notations. Denote by Γ_{ij} the edge which is common to two adjacent

substructures Ω_i and Ω_j and let $\Omega_{ij} = \Omega_i \cup \Gamma_{ij} \cup \Omega_j$. Then $\{\Omega_{ij}\}$ forms the basic decomposition of Ω, on which we will define our projections and this will eventually lead to an iterative substructuring algorithm. For simplicity, we denote $\Omega_{00} = \Omega$ and

$$\Lambda^E = \{(i,j) \mid x_i, x_j \in \Lambda^H, x_i, x_j \text{ adjacent, or } i = j = 0\}.$$

Associated with each Ω_{ij}, $(i,j) \in \Lambda^E$, we define a finite element subspace

$$V_{ij}^h \equiv H_0^1(\Omega_{ij}) \cap V^h, \text{ for } (i,j) \neq (0,0).$$

For $(i,j) = (0,0)$, we set $V_{00}^h = V^H$. It is easy to verify that

$$V^h = \sum_{(i,j)\in\Lambda^E} V_{ij}^h.$$

We note that there is a big difference between the decomposition $\{\Omega_i'\}$ and $\{\Omega_{ij}\}$. In the first case, we assume that the overlap is of order H, therefore for each node $x_i \in \Lambda^H$, there is a ball $O(x_i)$ with diameter of order H, centered at x_i, such that $O(x_i) \subset \Omega_k'$ for all Ω_k with x_i as one of its vertex. It follows from this property that we were able to use a local averaging technique to obtain to an optimal estimate. In the second case, this argument can not be carried out because the overlap is smaller, in the sense that no vertex of Λ^H belongs to any of the $\Omega_{i,j}$, $(i,j) \neq (0,0)$.

We denote the projection from V^h to V_{ij}^h with respect to the bilinear form $B(\cdot,\cdot)$ by P_{ij}^B, for $(i,j) \in \Lambda^E$.

P^B is defined, as usual, as the sum of the projections

$$P^B = \sum_{(i,j)\in\Lambda^E} P_{ij}^B.$$

This is the main operator we shall study in the following sections.

2. The basic algorithm and the main result

In this section, we present our basic algorithm and the corresponding convergence rate estimate.

Let $\mathbf{u}^h$ be the Galerkin approximation of the solution. By taking

$$\mathbf{b} \;=\; P^B \mathbf{u}^h \;=\; \sum_{(i,j)\in\Lambda^E} P^B_{ij}\mathbf{u}^h.$$

We can define our derived equation with respect to the bilinear form $B(\cdot,\cdot)$ and decomposition $\{V_{ij}\}$ as

$$P^B \mathbf{u}^h \;=\; \mathbf{b}.$$

Next, we will give one of the main theorem of this chapter, which estimates the conditioning of the operator P^B. Compared with the subspaces used in the previous section for AgM, we use less overlap here. This is reflected in the poorer bounds on the operator P^B.

Theorem 2.2 *(1) There exists a constant $C_p > 0$, independent of H and h, such that*

$$\|P^B u^h\|_A \le C_p \|u^h\|_A, \;\; \forall u^h \in V^h.$$

(2) There exists a constant $c > 0$, independent of H and h, such that

$$\|P^B u^h\|_A \ge c(1+\log(H/h))^{-2}\|u^h\|_A, \;\; \forall u^h \in V^h.$$

(3) There exist a constant $H_0 > 0$, independent of H and h, and a constant $c(H_0) > 0$, such that, for $H < H_0$,

$$(u^h, P^B u^h)_A \ge c_p(H_0, H, h)(u^h, u^h)_A, \; \forall u^h \in V^h,$$

where

$$c_p(H_0, H, h) \;=\; c(H_0)(1+\log(H/h))^{-2} > 0.$$

A proof will be given in the next section.

Remarks: (a) If $B(\cdot,\cdot)$ is selfadjoint, i.e. $A(\cdot,\cdot) = B(\cdot,\cdot)$, then the operator P^B is symmetric with respect to $A(\cdot,\cdot)$. Therefore, the generalized conjugate gradient method can be used. Theorem 2.2 part (1) and (2) give the convergence rate estimate.

(b) A remarkable fact about the algorithm mentioned in (a) is that if we reduce each subproblem P_{ij}^B to its Schur complement on Γ_{ij} and then replace the Schur complement by the square root of discrete one-dimensional Laplacian, denoted by $l_0^{1/2}$, then we obtain one of the Bramble, Pasciak and Schatz's algorithm [4]; cf.,M. Dryja and O. Widlund [12].

(c) In the early stage of the development of domain decomposition methods, a lot of algorithms were discussed for two subregion problems, which do not have very much interest for practical applications on multiprocessor machines. Among those algorithms were the Neumann-Dirichlet algorithm [2], modified Schur complement method [20]. However, we can apply these two region solvers as preconditioners for the semilocal problems represented by P_{ij}^B and obtain good algorithms for multiprocessor systems.

(d) In general, P^B is nonsymmetric but positive definite. the GMRES method is a generally applicable iterative method for this class of problems. Part (3) shows that the GMRES method converge in the $(\cdot,\cdot)_A$ inner product provided that the coarse mesh size is fine enough. The rate of convergence depends mildly on the mesh parameters H and h.

(e) This theorem holds only in $2D$.

2.3.2 The condition number estimates

We first quote a lemma, which plays an important role in the traditional theory for iterative substructuring algorithm. Variations of this result,

which dates back at least to 1966, are given in a number of papers, see e.g. Bramble [3], Bramble, Pasciak and Schatz [4] or Yserentant [29].

Lemma 2.4 *Let α be any value of $u^h(x) \in V^h$, with $x \in \Omega_i$, then*

$$\|u^h - \alpha\|^2_{L^\infty(\Omega_i)} \leq C(1 + \log(H/h)) \mid u^h \mid^2_{H^1(\Omega_i)}$$

Remark: This result holds only in $2D$. The $1 + \log(H/h)$ factor which appears in Theorem 2.2 is introduced here.

The next lemma due to M. Dryja and O. Widlund [12], provides a decomposition of all functions in V^h into a sum of functions defined in subspaces. This decomposition is not uniformly bounded as there is less overlap, but the bound depends only mildly on the size parameters h and H.

Lemma 2.5 *For any $u^h \in V^h$, there exist $u^h_{ij} \in V^h_{ij}$, $(i,j) \in \Lambda^E$, such that*

$$u^h = \sum_{(i,j)\in\Lambda^E} u^h_{ij},$$

and there exists a constant C, independent of u^h, h and H, such that

$$\sum_{(i,j)\in\Lambda^E} \|u^h_{ij}\|^2_{A(\Omega_{ij})} \leq C(1 + \log(H/h))^2 \|u^h\|^2_A$$

Proof:(See Dryja and Widlund, [12])

Before we prove theorem 2.2 part (3), we need to formulate a lemma, which has already been proved in the previous section. Since the notations are a little bit different, we rewrite the lemma here without proof.

Lemma 2.6 *There exists a constant $C > 0$, such that*

$$(a) \quad \mid N(P^B_{00}u^h, u^h - P^B_{00}u^h) \mid \leq CH^\gamma \parallel P^B_{00}u^h \parallel_A \parallel u^h \parallel_A, \quad \forall u^h \in V^h$$

$$(b) \quad | N(P_{ij}^B u^h, u^h - P_{ij}^B u^h) | \leq CH \| P_{ij}^B u^h \|_{A(\Omega_{i,j})} \| u^h \|_{A(\Omega_{i,j})}, \quad \forall u^h \in V^h$$
$$for\ (i,j) \neq (0,0) \in \Lambda^E.$$

Remark: By using the same argument as in the Remark after Lemma 2.3, we can show that if the coarse mesh size is small enough, we have

$$| N(u^h, P^B u^h) | \leq \delta | B(u^h, P^B u^h) |, \quad \forall u^h \in V^h,$$

where $0 < \delta < 1$. Therefore, the assumption for Theorem 1.3 is established.

Chapter 3

Methods for parabolic convection-diffusion problems

In this chapter, we apply the $AS\!M$ and $IS\!M$ to the parabolic convection-diffusion problems. A modified $AS\!M$ is also introduced. We consider the linear systems of equations that arise when using implicit schemes to approximate parabolic problems. If we consider the discrete parabolic problem at a fixed time level, it is equivalent to an elliptic problem with an extra time step parameter. The central mathematical question is to estimate how the convergence rate depends on the time step parameter, especially in the case that this parameter is relatively large, as well as the space mesh parameters. Related works can be found in [21], [15].

The outline of this chapter is as follows. In section 1, we present the parabolic problem. In section 2, we study the additive Schwarz algorithm. In section 3, we study a modified additive Schwarz algorithm, which works well only for parabolic problems. Finally in section 4, we study an iterative substructuring method.

3.1 A parabolic convection-diffusion problem

We consider the following parabolic convection-diffusion problem: Find $u(x,t)$, such that

$$\begin{cases} \partial u/\partial t + Lu = f, & \text{in } \Omega \times [0,T], \\ u = 0 & \text{on } \partial\Omega \times [0,T], \\ u(x,0) = u_0(x) & \text{in } \Omega, \end{cases} \tag{3.1}$$

where L is a strongly elliptic operator defined in Chapter 2. $\Omega \subset R^d$ is a polygonal domain with boundary $\partial\Omega$.

Let us consider the corresponding variational problem: Find $u(x,t) \in H_0^1(\Omega)$, $t \in [0,T]$, $u(x,0) = u_0(x)$ in Ω, such that

$$(\frac{\partial u}{\partial t}, v) + B(u,v) = (f,v), \quad \forall v \in H_0^1(\Omega),$$

where the bilinear form $B(\cdot,\cdot)$ and the linear functional (f,v) are the same as in Chapter 2.

The existence and uniqueness of the solution of the variational parabolic convection-diffusion equation are well understood, see [19].

We use two types of time discretization, namely, a backward Euler scheme and an implicit Crank-Nicolson scheme.

Let Δt_n be the n^{th} time step, M is the number of steps and $\sum_{n=1}^{M} \Delta t_n = T$. For the first scheme, the time discrete problem is

$$(\frac{u^n - u^{n-1}}{\Delta t_n}, v) + B(u^n, v) = (f,v), \quad \forall v \in H_0^1(\Omega),$$

with $u^0(x,t) = u_0(x)$ and $n = 1, \cdots, M$.

For the second scheme,

$$(\frac{u^n - u^{n-1}}{\Delta t_n}, v) \ + \ B(\frac{u^n + u^{n-1}}{2}, v) \ = \ (f, v), \quad \forall v \in H_0^1(\Omega),$$

with $u^0(x,t) = u_0(x)$ and $n = 1, \cdots, M$.

To find the n^{th} level solution, both schemes lead to the following variational problem: Given function $g_{n-1} \in L^2(\Omega)$, find $w \in H_0^1(\Omega)$, such that

$$(w, v) \ + \ \tau B(w, v) \ = \ (g_{n-1}, v), \quad \forall v \in H_0^1(\Omega). \tag{3.2}$$

We call τ the time step parameter. It is easy to verify that for the backward Euler scheme

$$\begin{aligned} w \quad &= \quad u^n - u^{n-1}, \\ \tau \quad &= \quad \Delta t_n, \\ (g_{n-1}, v) \quad &= \quad \tau((f, v) \ - \ B(u^{n-1}, v)) \end{aligned}$$

and for the Crank-Nicolson scheme

$$\begin{aligned} w \quad &= \quad u^n - u^{n-1}, \\ \tau \quad &= \quad \Delta t_n/2, \\ (g_{n-1}, v) \quad &= \quad \tau(2(f, v) \ - \ B(u^{n-1}, v)). \end{aligned}$$

The stability of both schemes is well understood, see [19]. In this chapter we focus on the study of fast iterative algorithm for solving the resulting large linear systems at each time step. The variational problem (3.2) is the main subject of our further study.

To simplify the notations, we introduce a bilinear form

$$D_\tau(w, v) \ = \ (w, v) \ + \ \tau B(w, v).$$

In general, $D_\tau(\cdot, \cdot)$ is not symmetric. For technical reasons it is convenient to separate the symmetric and the skewsymmetric part of this bilinear form. We therefore introduce the bilinear forms as

$$A_\tau(u, v) \ = \ 1/2(D_\tau(u, v) \ + \ D_\tau(v, u)),$$

which is the symmetric part, and

$$N_\tau(u,v) \;=\; 1/2(D_\tau(u,v) \;-\; D_\tau(v,u)),$$

which is the skewsymmetric part.

We assume that $N_\tau(\cdot,\cdot)$ is bounded in the following sense: There exists a constant $C > 0$, such that

$$\mid N_\tau(u,v) \mid \leq C\tau \|u\|_{H_0^1(\Omega)} \|v\|_{L^2(\Omega)}, \quad \forall u, v \in H_0^1(\Omega).$$

Before we study the finite element solution of (3.2), we need to establish some bounds for the bilinear form $D_\tau(w,v)$. Define the τ-norm as the square root of

$$\| \cdot \|_{\tau(\Omega)}^2 = \| \cdot \|_{L^2(\Omega)}^2 \; + \tau \; \| \cdot \|_{H_0^1(\Omega)}^2 .$$

The following lemma gives the boundness and positive definiteness of the bilinear form $D_\tau(\cdot,\cdot)$.

Lemma 3.1 *There exist positive constants c_1, c_2, c_3 and c_4, which are independent of τ, such that*

(1) $\mid D_\tau(w,v) \mid \leq c_1 \parallel w \parallel_{\tau(\Omega)} \parallel v \parallel_{\tau(\Omega)}, \quad \forall w, v \in H_0^1(\Omega).$

(2) $D_\tau(w,w) \geq c_2 \parallel w \parallel_{\tau(\Omega)}^2, \quad \forall w \in H_0^1(\Omega).$

(3) $\mid A_\tau(w,v) \mid \leq c_3 \parallel w \parallel_{\tau(\Omega)} \parallel v \parallel_{\tau(\Omega)}, \quad \forall w, v \in H_0^1(\Omega).$

(4) $A_\tau(w,w) \geq c_4 \parallel w \parallel_{\tau(\Omega)}^2, \quad \forall w \in H_0^1(\Omega).$

Proof: (1) Because the boundedness of $B(w,v)$, we have

$$\mid D_\tau(w,v) \mid \leq \parallel w \parallel_{L^2(\Omega)} \parallel v \parallel_{L^2(\Omega)} \; + c\tau \; \parallel w \parallel_{H_0^1(\Omega)} \parallel v \parallel_{H_0^1(\Omega)} .$$

The estimate (1) follows from Cauchy's inequality.

The estimate (2) follows from the ellipticity of $B(\cdot,\cdot)$.

The estimate (3) is a corollary of (1), and the estimate (4) is a corollary of (2). □

If we denote $\|u\|_{A_\tau} = \sqrt{A_\tau(u,u)}$, then we know from the above lemma that the A_τ-norm is equivalent to the τ-norm. We shall use A_τ-norm in our discussion.

To obtain a fully discrete problem, we discretize equation (3.2) in space by using Galerkin's finite element method in the subspace $V^h \subset H_0^1(\Omega)$. The finite dimensional approximation of equation (3.2) reads as: At each step n, find $w^h \in V^h$, such that

$$D_\tau(w^h, v^h) \;=\; (g_{n-1}, v^h), \;\; \forall v^h \in V^h. \tag{3.3}$$

This equation is the main subject of this chapter. We shall use three methods to solve it.

Based on the $H^{1+\gamma}(\Omega)$ regularity assumption on $B(u,v)$, see (2.2), the equation (3.3) satisfies the following the regularity result.

Lemma 3.2 *For any $g \in L^2(\Omega)$, the equation*

$$D_\tau(v, w) \;=\; (g, v), \;\; \forall v \in H_0^1(\Omega)$$

has solution $w \in H^{1+\gamma}(\Omega) \cap H_0^1(\Omega)$, and moreover, there exists a constant $C > 0$, independent of τ, w and g, such that

$$| \, w \, |_{H^{1+\gamma}(\Omega)} \leq C/\tau \|g\|_{L^2(\Omega)}.$$

Proof: Consider the equality

$$(w, w) + \tau B(w, w) = (g, w).$$

Since $B(w,w) \geq 0$, we obtain that

$$\|w\|_{L^2(\Omega)} \leq \|g\|_{L^2(\Omega)},$$

which implies that

$$\|w - g\|_{L^2(\Omega)} \leq 2\|g\|_{L^2(\Omega)}.$$

On the other hand, we have

$$B(v, w) = ((g - w)/\tau, v), \quad \forall v \in H_0^1(\Omega).$$

By the assumption (2.2), there exists a constant $C > 0$ such that

$$| w |_{H^{1+\gamma}(\Omega)} \leq C\|(g - w)/\tau\|_{L^2(\Omega)}.$$

□

3.2 Additive Schwarz method for the problems in R^2 and R^3

3.2.1 An algorithm and the main results

In this section, we propose an additive Schwarz algorithm for the finite element equation (3.3).

As shown in Chapter 2, the finite element space V^h can be decomposed into the sum of a coarse mesh function space and a number of function spaces which are supported only in subregions Ω_i', i.e.

$$V^h = V_0^h + V_1^h + \cdots + V_N^h.$$

We denote by $P_i^{D_\tau}$ to be the projection from V^h to V_i^h with respect to the bilinear form $D_\tau(\cdot, \cdot)$ and let $P^{D_\tau} = P_0^{D_\tau} + \cdots + P_N^{D_\tau}$.

The additive Schwarz algorithm for equation (3.3) can be stated as: At each step n, find $w^h \in V^h$, such that

$$\begin{aligned} P^{D_\tau} w^h &= g_{n-1}', \\ u^n &= u^{n-1} + w^h, \end{aligned} \tag{3.4}$$

which is the derived Galerkin finite element equation with respect to the bilinear form $D_\tau(\cdot,\cdot)$ and the decomposition $\{V_i\}$. We will prove that this system is uniformly well conditioned. More precisely, the condition number of P^{D_τ} will not change if we (1) refine the fine mesh size h to increase the accuracy; (2) refine the coarse mesh size H so that more processors can be used; (3) increase the time step τ.

For the conditioning of the operator P^{D_τ}, we have

Theorem 3.1 *(1) There exists a constant $C_p > 0$, independent of H, h, τ, such that*

$$\|P^{D_\tau} u^h\|_{A_\tau} \leq C_p \|u^h\|_{A_\tau}, \quad \forall u^h \in V^h.$$

(2) There exists a constant $c > 0$, independent of H, h, τ, such that

$$\|P^{D_\tau} u^h\|_{A_\tau} \geq c \|u^h\|_{A_\tau}, \quad \forall u^h \in V^h.$$

(3) If $c_{H,\tau} = \max\{H, H^\gamma \sqrt{H^2/\tau + 1}\}$ is small enough, i.e. $0 < c_{H,\tau} \leq \tilde{c}_0$, then there exists a constant $c_p(\tilde{c}_0) > 0$, such that

$$(u^h, P^{D_\tau} u^h)_{A_\tau} \geq c_p(\tilde{c}_0)(u^h, u^h)_{A_\tau}, \quad \forall u^h \in V^h.$$

A proof is given in the next section.

Remarks: (a) In the case that the first order terms in L vanish, the problem is selfadjoint. It can be seen easily that the operator P^{D_τ} is symmetric with respect to A_τ-norm. The standard conjugate gradient method in A_τ-norm can therefore be used. Theorem 3.1 shows that this P^{D_τ} linear system is optimal for the conjugate gradient method in the sense that the rate of convergence does not depend on the mesh parameters H and h, nor on the time step size τ.

(b) Since, in general, (3.4) is a nonsymmetric but positive definite system, we use the GMRES method to solve it. The inner product we use is $[\cdot,\cdot] = (\cdot,\cdot)_{A_\tau}$.

(c) In general, $\tilde{c}_0$ depends on the coefficients of the first order terms in L, the ellipticity constant of D_τ , the bounds on D_τ and also the geometry of the domain Ω. We do not have a explicit relation between $\tilde{c}_0$ and the skewsymmetric coefficients in L. From the lower bound proof, which will be given in the next section, we know that as the skewsymmetric coefficients in L increase, $\tilde{c}_0$ decreases.

3.2.2 Proof of the theorem

In this section, we prove the theorem stated in the previous section. Most of the techniques are similar to those that were used for the time independent problems. To complete the proof, we need only to show that all the assumption of Theorem 1.3 hold, and in addition that all the constants that appear in the abstract Theorem 1.3 are independent of the mesh parameters H, h and τ as stated in Theorem 3.1.

We start with some lemmas which contain most of the basic results.

The following lemma is well known. It says that in the finite element space V^h the continuous L^2 norm is equivalent to the discrete L^2 norm. We do not include the proof.

Lemma 3.3 *There exist two constants $c_1 > 0$ and $c_2 > 0$, which depend only on the shape regularity of the finite element subdivision of Ω, such that*

$$c_1 h^d \sum_{x_i \in \Lambda^h} (u^h(x_i))^2 \leq \| u^h \|^2_{L^2(\Omega)} \leq c_2 h^d \sum_{x_i \in \Lambda^h} (u^h(x_i))^2, \ \forall u^h \in V^h.$$

Here d is the dimension of the space. The statement is also true if we replace V^h by V^H and h by H.

Recall that when we considered the ASM for the time independent problems, a $H_0^1(\Omega)$ decomposition lemma played a very important role in the condition number estimate. Next, we prove that the same lemma holds in $L^2(\Omega)$ norm. By combining the estimates in $H_0^1(\Omega)$ and $L^2(\Omega)$, we eventually prove a decomposition lemma in the $\|\cdot\|_{A_\tau}$ norm, which will be used for the time dependent problems.

Lemma 3.4 *$\forall u^h \in V^h$, there exist $u_i^h \in V_i^h$, $i = 0, 1, \cdots, N$, such that*

$$u^h = \sum_{i=0}^{N} u_i^h$$

and, moreover, there exists a constant $C > 0$, such that

$$\sum_{i=0}^{N} \| u_i^h \|^2_{L^2(\Omega)} \leq C \| u^h \|^2_{L^2(\Omega)},$$

where C is independent of h, H and u^h.

Proof: The construction of u_i^h, $i = 0, 1, \cdots, N$ is the same as in the $H_0^1(\Omega)$ case. Let $\hat{I}_H$ be L^2 projection operator into the coarse mesh space defined in Chapter 2. We take $u_0^h = \hat{I}_H u^h$, $w^h = u^h - \hat{I}_H u^h$, and then set $u_i^h = I_h(\theta_i(w^h))$, where the partition of unity $\{\theta_i\}$ is the same as in Chapter 2.

Because $\hat{I}_H u^h$ is the L^2 projection, we have

$$\| \hat{I}_H u^h \|^2_{L^2(\Omega)} \leq \| u^h \|^2_{L^2(\Omega)} . \tag{3.5}$$

Next, we prove the $L^2(\Omega)$ boundness for the other u_i^h. By using Lemma 3.3 for each Ω_i',

$$\sum_{i=1}^{N} \| I^h(\theta_i w^h) \|^2_{L^2(\Omega_i')} \leq C \sum_{i=1}^{N} \Big(\sum_{x_j \in \Lambda^h \cap \Omega_i'} ((\theta_i w^h)(x_j))^2 \Big) h^2.$$

Since $\mid \theta_i \mid \leq 1$, $i = 1, \cdots, N$, and $\{\Omega_i'\}$ is a finite covering of Ω, the right hand side can be bounded by

$$\sum_{x_i \in \Lambda^h} (w^h(x_i))^2 h^2.$$

Using Lemma 3.3 again, we obtain

$$\sum_{i=1}^{N} \| u_i^h \|_{L^2(\Omega_i)}^2 \leq C \| w^h \|_{L^2(\Omega)}^2 \leq C \| u^h \|_{L^2(\Omega)}^2 . \tag{3.6}$$

Combining the estimates (3.5) and (3.6) proves this lemma.□

The decomposition lemma for the $\| \cdot \|_{\tau(\Omega)}$ norm follows immediately from Lemma 3.4 and the $H_0^1(\Omega)$ decomposition Lemma 2.2. We simply state it as follows:

Lemma 3.5 *$\forall u^h \in V^h$, there exist $u_i^h \in V_i^h$, $i = 0, 1, \cdots, N$, such that*

$$u^h = \sum_{i=0}^{N} u_i^h$$

and, moreover, there exists a constant $C > 0$, such that

$$\sum_{i=0}^{N} \| u_i^h \|_{A_\tau(\Omega_i')}^2 \leq C \| u^h \|_{A_\tau}^2,$$

where C is independent of h, H and u^h.

Lemma 3.6 *For any $w \in H_0^1(\Omega) \cap H^{1+\gamma}(\Omega)$, there exists a $w^H \in V^H$, such that*

$$\| w - w^H \|_{\tau(\Omega)} \leq C H^\gamma \sqrt{H^2 + \tau} \mid w \mid_{H^{1+\gamma}(\Omega)},$$

where C is independent of τ, w and H.

Proof: For any given w, let w^H be the solution of

$$a(w^H, v) = a(w, v), \quad \forall v \in V^H,$$

where $a(u, v) = \int_\Omega \nabla u \nabla v d\Omega$. By the classical finite element approximation theory, we know that

$$\| w - w^H \|_{H_0^1(\Omega)} \leq C H^\gamma \mid w \mid_{H^{1+\gamma}(\Omega)},$$

$$\| w - w^H \|_{L^2(\Omega)} \leq C H^{1+\gamma} \mid w \mid_{H^{1+\gamma}(\Omega)}.$$

Hence, we have

$$\| w - w^H \|_{\tau(\Omega)} \leq C H^\gamma \sqrt{H^2 + \tau} \mid w \mid_{H^{1+\gamma}(\Omega)}.$$

□

Recall that $P_0^{D_\tau}$ is the projection from V^h into V^H with respect to the $D_\tau(\cdot, \cdot)$ norm, thus a H^1 projection in some sense. We can therefore expect the L^2 approximation to contribute an extra factor H. The $L^2(\Omega)$ estimate is given in the following lemma.

Lemma 3.7 *There exists a constant $C > 0$, which is independent of both H and τ, such that*

$$\| u^h - P_0^{D_\tau} u^h \|_{L^2(\Omega)} \leq C H^\gamma (\sqrt{H^2 + \tau})/\tau \| u^h \|_{\tau(\Omega)}, \quad \forall u^h \in V^h.$$

Proof: We first establish a bound for the τ-norm. It is easy to verify that

$$D_\tau(u^h - P_0^{D_\tau} u^h, u^h - P_0^{D_\tau} u^h) = D_\tau(u^h - P_0^{D_\tau} u^h, u^h).$$

Using Lemma 3.1 for both sides, we get the estimate

$$\| u^h - P_0^{D_\tau} u^h \|_{\tau(\Omega)} \leq C \| u^h \|_{\tau(\Omega)}.$$

The $H_0^1(\Omega)$ (or τ-norm) estimate and the Nitsche trick give the $L^2(\Omega)$ estimate. We have

$$\|u^h - P_0^{D_\tau} u^h\|_{L^2(\Omega)} = \sup_{\|v\|_{L^2(\Omega)} \neq 0} \frac{(u^h - P_0^{D_\tau} u^h, v)}{\|v\|_{L^2(\Omega)}}.$$

For any fixed $v \in L^2(\Omega)$, we form an auxiliary variational problem: Find $w \in H^{1+\gamma}(\Omega) \cap H_0^1(\Omega)$, such that

$$D_\tau(\phi, w) = (\phi, v), \quad \forall \phi \in H_0^1(\Omega).$$

By Lemma 3.2, we have $|w|_{H^{1+\gamma}(\Omega)} \leq C/\tau \|v\|_{L^2(\Omega)}$.

We take $\phi = u^h - P_0^{D_\tau} u^h \in H_0^1(\Omega)$; then

$$(u^h - P_0^{D_\tau} u^h, v) = D_\tau(u^h - P_0^{D_\tau} u^h, w).$$

Take w^H to be the V^H approximation of w obtained in Lemma 3.6. Since $w^H \in V^H$, we have

$$\begin{aligned} | D_\tau(u^h - P_0^{D_\tau} u^h, w) | &= | D_\tau(u^h - P_0^{D_\tau} u^h, w - w^H) | \\ &\leq C\|u^h - P_0^{D_\tau} u^h\|_{\tau(\Omega)} \|w - w^H\|_{\tau(\Omega)} \\ &\leq CH^\gamma \sqrt{H^2 + \tau} \|u^h\|_{\tau(\Omega)} \mid w \mid_{H^{1+\gamma}(\Omega)}. \end{aligned}$$

Combining the above results, we have

$$\|u^h - P_0^{D_\tau} u^h\|_{L^2(\Omega)} \leq CH^\gamma(\sqrt{H^2 + \tau})/\tau \|u^h\|_{\tau(\Omega)},$$

which is the desired result. □

In the next lemma, we estimate the energy contributed by the skewsymmetric part $N_\tau(\cdot, \cdot)$. We show that it is in fact a lower order term compared with the symmetric part, and hence can be controlled if the coarse mesh size is fine enough.

Lemma 3.8 *If* $\max\{H, H^\gamma\sqrt{H^2/\tau+1}\}$ *is small enough, there exists a constant* $0<\delta<1$, *such that*

$$| N_\tau(u^h, P^{D_\tau}u^h) | \leq \delta D_\tau(u^h, P^{D_\tau}u^h), \quad \forall u^h \in V^h.$$

Proof: Since $N_\tau(P_i^{D_\tau}u^h, P_i^{D_\tau}u^h) = 0$, it is easy to verify that

$$| N_\tau(u^h, P^{D_\tau}u^h) | \leq \sum_{i=0}^{N} | N_\tau(P_i^{D_\tau}u^h, u^h - P_i^{D_\tau}u^h) | .$$

Therefore, we need only to estimate the right hand side of the above inequality. Since the coarse mesh projection $P_0^{D_\tau}$ is special, we will consider it separately using the result of Lemma 3.7.

(1) If $i=0$, by Lemma 3.7

$$\begin{aligned} | N_\tau(P_0^{D_\tau}u^h, u^h - P_0^{D_\tau}u^h) | &\leq C\tau\|P_0^{D_\tau}u^h\|_{H_0^1(\Omega)}\|u^h - P_0^{D_\tau}u^h\|_{L^2(\Omega)} \\ &\leq CH^\gamma\sqrt{H^2+\tau}\|P_0^{D_\tau}u^h\|_{H_0^1(\Omega)} \cdot \|u^h\|_{\tau(\Omega)}. \end{aligned}$$

It is easy to see that

$$\sqrt{\tau}\|v\|_{H_0^1(\Omega)} \leq \|v\|_{\tau(\Omega)}.$$

Hence,

$$| N_\tau(P_0^{D_\tau}u^h, u^h - P_0^{D_\tau}u^h) | \leq CH^\gamma\sqrt{H^2/\tau+1}\|P_0^{D_\tau}u^h\|_{\tau(\Omega)}\|u^h\|_{\tau(\Omega)}.$$

From the definition of $P_0^{D_\tau}$, we have

$$D_\tau(P_0^{D_\tau}u^h, P_0^{D_\tau}u^h) = D_\tau(u^h, P_0^{D_\tau}u^h).$$

If we apply Lemma 3.1 for both sides, we obtain

$$\|P_0^{D_\tau}u^h\|_{\tau(\Omega)} \leq C\|u^h\|_{\tau(\Omega)}.$$

Therefore, the first term can be bounded as follows

$$| N_\tau(P_0^{D_\tau}u^h, u^h - P_0^{D_\tau}u^h) | \leq CH^\gamma\sqrt{H^2/\tau + 1}\|u^h\|^2_{\tau(\Omega)}.$$

(2)If $i \neq 0$,

$$\begin{aligned} | N_\tau(P_i^{D_\tau}u^h, u^h - P_i^{D_\tau}u^h) | &\leq C\tau\|P_i^{D_\tau}u^h\|_{L^2(\Omega_i')}(\|u^h\|_{H^1(\Omega_i')} + \|P_i^{D_\tau}u^h\|_{H_0^1(\Omega_i')}) \\ &\leq C\sqrt{\tau}\|P_i^{D_\tau}u^h\|_{L^2(\Omega_i')}(\|u^h\|_{\tau(\Omega_i')} + \|P_i^{D_\tau}u^h\|_{\tau(\Omega_i')}). \end{aligned}$$

The first factor can be estimated by using Poincaré's inequality. Since $P_i^{D_\tau}u^h \in H_0^1(\Omega_i')$, the diameter of Ω_i' of order H, and we have

$$\|P_i^{D_\tau}u^h\|_{L^2(\Omega_i')} \leq CH\|P_i^{D_\tau}u^h\|_{H_0^1(\Omega_i')}.$$

By the definition of $P_i^{D_\tau}$,

$$D_\tau(P_i^{D_\tau}u^h, P_i^{D_\tau}u^h) = D_\tau(u^h, P_i^{D_\tau}u^h).$$

Using Lemma 3.1 for both sides, we obtain

$$\|P_i^{D_\tau}u^h\|_{\tau(\Omega_i')} \leq C\|u^h\|_{\tau(\Omega_i')}.$$

Combining these inequalities, we obtain

$$| N_\tau(P_i^{D_\tau}u^h, u^h - P_i^{D_\tau}u^h) | \leq CH\|u^h\|^2_{\tau(\Omega_i')}.$$

Putting the results in (1) and (2) together, we have

$$| N_\tau(P^{D_\tau}u^h, u^h) | \leq C\max\{H, H^\gamma\sqrt{H^2/\tau + 1}\}\sum_{i=0}^{N}\|u^h\|^2_{\tau(\Omega_i')}, \qquad (3.7)$$

which also holds in A_τ-norm. It is easy to see that

$$\sum_{i=0}^{N}\|P_i^{D_\tau}u^h\|^2_{A_\tau(\Omega_i')} = D_\tau(u^h, P^{D_\tau}u^h).$$

By using the decomposition lemma, we have

$$\begin{aligned}\|u^h\|^2_{A_\tau} &= \textstyle\sum_{i=0}^N D_\tau(P_i^{D_\tau}u^h, u_i^h)\\ &\leq C\textstyle\sum_{i=0}^N \|P_i^{D_\tau}u^h\|_{A_\tau(\Omega_i')}\|u_i^h\|_{A_\tau(\Omega_i')}\\ &\leq C\sqrt{\textstyle\sum_{i=0}^N \|u_i^h\|^2_{A_\tau(\Omega_i')}}\sqrt{\textstyle\sum_{i=0}^N \|P_i^{D_\tau}u^h\|^2_{A_\tau(\Omega_i')}}.\end{aligned}$$

Therefore, we obtain

$$\|u^h\|^2_{A_\tau} \leq C\sum_{i=0}^N \|P_i^{D_\tau}u^h\|^2_{A_\tau(\Omega_i')}.$$

Hence

$$\|u^h\|^2_{A_\tau} \leq CD_\tau(u^h, P^{D_\tau}u^h). \tag{3.8}$$

We complete the proof by combining estimates (3.7) and (3.8). □

The proof of Theorem 3.1 can be accomplished by using Theorem 1.3 and the lemmas in this section.

3.3 Modified additive Schwarz method for the problems in R^2

3.3.1 An algorithm and the main results

In this section, we propose a modified version of $AS/$ by dropping the coarse mesh space V_0^h, which represents the global information transportation. We show that in some situations the global function space is not necessary for fast convergence. This is true only for parabolic equations.

Let us define $\tilde{P}^{D_\tau} = P_1^{D_\tau} + \cdots + P_N^{D_\tau}$, where the $P_i^{D_\tau}$ are the same as in the previous section. Note that we excluded $P_0^{D_\tau}$.

We use the same algorithm, presented in the previous section, with P^{D_τ} replaced by $\tilde{P}^{D_\tau}$. About the conditioning of the operator $\tilde{P}^{D_\tau}$, we have the following theorem.

Theorem 3.2 *(1) There exists a constant $C_{\tilde{P}} > 0$, such that*

$$\|\tilde{P}^{D_\tau} u^h\|_{A_\tau} \leq C_{\tilde{P}} \|u^h\|_{A_\tau}, \quad \forall u^h \in V^h.$$

(2) There exists a constant $c > 0$, such that

$$\|\tilde{P}^{D_\tau} u^h\|_{A_\tau} \geq c(1 + \tau/H^2)^{-1} \|u^h\|_{A_\tau}, \quad \forall u^h \in V^h.$$

(3) If $c_{H,\tau} = H(1 + \tau/H^2)$ is small enough, i.e. $0 < c_{H,\tau} \leq \tilde{c}_0$, then there exists a constant $c_{\tilde{P}}(\tilde{c}_0) > 0$, such that

$$(u^h, \tilde{P}^{D_\tau} u^h)_{A_\tau} \geq c_{\tilde{P}}(\tilde{c}_0)(u^h, u^h)_{A_\tau}, \quad \forall u^h \in V^h.$$

A proof is given in the next section.

Remarks: (a) For symmetric problems, part (1) and (2) of this theorem shows that if the factor τ/H^2 is small, the dropping of the coarse mesh space does not lead to slow convergence. This suggests we might use the modified AsM in the case that we have a relatively small time step or large substructures.

For nonsymmetric problems, in order to obtain the fast convergence, we need both H and τ/H^2 to be small. This implies that we cannot choose τ and H independently.

(b) This theorem is true only in R^2. In higher dimension, the fine mesh size h enters our bounds.

3.3.2 Proof of the theorem

The proof of Theorem 3.2 can be accomplished by using the results in Theorem 1.3, a new decomposition lemma given below and the new estimate of the skewsymmetric part. We begin with the decomposition lemma.

Lemma 3.9 *$\forall u^h \in V^h$, there exist $u_i^h \in V_i^h$, $i = 1, \cdots, N$, such that*

$$u^h = \sum_{i=1}^{N} u_i^h$$

and, moreover, there exists a constant $C > 0$, such that

$$\sum_{i=1}^{N} \| u_i^h \|^2_{A_\tau(\Omega_i')} \leq C(1 + \tau/H^2) \| u^h \|^2_{A_\tau},$$

where C is independent of h, H and τ.

Proof: We first construct the decomposition, then we do the estimates in both $H_0^1(\Omega)$ and $L^2(\Omega)$ norm.

Let $\{\theta_i, i = 1, \cdots, N\}$ be the partition of unity defined in Chapter 2. Denote

$$u_i^h = I_h(\theta_i u^h), \quad i = 1, \cdots, N.$$

For each substructure Ω_i', we have

$$| u_i^h |^2_{H_0^1(\Omega_i')} = \sum((\theta_i u^h)(x_l) - (\theta_i u^h)(x_m))^2,$$

where the sum is taken over all adjacent pairs of nodal points x_l and x_m in Ω_i'. Let $K \subset \Omega_i'$ be a single element and x_l, $x_m \in K$. Denote

$$\bar{\theta}_{ilm} = 1/2(\theta_i(x_l) + \theta_i(x_m)).$$

We then have

$$(\theta_i u^h)(x_l) - (\theta_i u^h)(x_m) =$$

$$(\theta_i(x_l) - \bar{\theta}_{ilm})u^h(x_l) - (\theta_i(x_m) - \bar{\theta}_{ilm})u^h(x_m) + \bar{\theta}_{ilm}(u^h(x_l) - u^h(x_m)),$$

which can be bounded from above by

$$C(h/H \max_K\{| u^h(x) |\} + | u^h(x_l) - u^h(x_m) |).$$

By squaring this estimate, using the triangle inequality and summing over all $K \subset \Omega_i'$, we obtain

$$\begin{aligned} | u_i^h |^2_{H_0^1(\Omega_i')} &\leq C(H^{-2} \textstyle\sum_{K \subset \Omega_i'} \max_K \{| u^h(x) |\}^2 \cdot h^2 + | u^h |^2_{H_0^1(\Omega_i')}) \\ &\leq C(H^{-2} \|u^h\|^2_{L^2(\Omega_i')} + | u^h |^2_{H_0^1(\Omega_i')}). \end{aligned}$$

Using Poincaré's inequality, we have

$$\|u_i^h\|^2_{L^2(\Omega_i')} \leq CH^2 | u_i^h |^2_{H_0^1(\Omega_i')} .$$

Therefore

$$\begin{aligned} \textstyle\sum_{i=1}^N \|u_i^h\|^2_{H_0^1(\Omega_i')} &\leq (1 + H^2) \textstyle\sum_{i=1}^N | u_i^h |^2_{H_0^1(\Omega_i')} \\ &\leq C(1 + H^2)(H^{-2} \|u^h\|^2_{L^2(\Omega)} + | u^h |^2_{H_0^1(\Omega)}) \\ &\leq CH^{-2} | u^h |^2_{H_0^1(\Omega)} . \end{aligned}$$

Hence, we obtain

$$\sum_{i=1}^N \|u_i^h\|^2_{H_0^1(\Omega_i')} \leq CH^{-2} \|u^h\|^2_{H_0^1(\Omega)}.$$

Combined with the L^2 result given in Lemma 3.4 and the definition of the τ-norm, we complete the proof. □

In the next lemma, we estimate the energy contributed by the skewsymmetric part $N_\tau(\cdot, \cdot)$.

Lemma 3.10 *There exists a constant $C > 0$, which is independent of h, H and τ, such that*

$$| N_\tau(u^h, \tilde{P}^{D_\tau} u^h) | \leq CH(1 + \tau/H^2) D_\tau(u^h, \tilde{P}^{D_\tau} u^h), \quad \forall u^h \in V^h.$$

The proof can be obtained by using the proof of Lemma 3.8 part (2) and replacing the decomposition Lemma 3.5 by Lemma 3.9.

3.4 Iterative substructuring method for the problems in R^2

3.4.1 An algorithm and the main results

1. Basic domain decomposition and projections

We assume that the decomposition of Ω is $\{\Omega_{ij},\ \ (i,j) \in \Lambda^E\}$, which is the same as the one introduced in Chapter 2. The corresponding V^h function space decomposition is also the same

$$V^h = \sum_{(i,j)\in\Lambda^E} V_{ij}^h$$

Denote by $P_{ij}^{D_\tau}$ to be the projection from V^h to subspace V_{ij}^h with respect to the bilinear form $D_\tau(\cdot,\cdot)$, which was defined in Chapter 2.

The mapping $P^{D_\tau} : V^h \longrightarrow V^h$ is defined as

$$P^{D_\tau} = \sum_{(i,j)\in\Lambda^E} P_{ij}^{D_\tau},$$

which is one of the main operators we shall deal with in this chapter.

2. The basic algorithm and the main results

In this section, we first introduce our iterative substructuring algorithm for the parabolic convection-diffusion equation. Then, we state the theorem concerning the rate of convergence of the algorithm. The proof of the theorem will be given in the next section.

Let us denote

$$\mathbf{b}^{D_\tau} = P^{D_\tau}\mathbf{u} = \sum_{(i,j)\in\Lambda^E} P_{ij}^{D_\tau}\mathbf{u}.$$

Our basic iterative substructuring algorithm for solving equation 3.3 is as follows:

Basic algorithm: Solve the following linear system of equations

$$P^{D_\tau}\mathbf{u} = \mathbf{b}^{D_\tau}$$

by the GMRES method using the inner product $[\cdot,\cdot] = (\cdot,\cdot)_{A_\tau}$.

We next present some bounds concerning the conditioning of the operator P^{D_τ}. The rate of convergence of the above algorithm can be estimated by using these bounds. Let us denote

$$\nu(h, H, \tau) = (1 + H^2/\tau)(1 + \log H/h).$$

Theorem 3.3 *(1) There exists a constant $C_p > 0$, independent of H, h, τ, such that*

$$\|P^{D_\tau} u^h\|_{A_\tau} \leq C_p \|u^h\|_{A_\tau}, \quad \forall u^h \in V^h.$$

(2) There exists a constant $c > 0$, independent of H, h, τ, such that

$$\|P^{D_\tau} u^h\|_{A_\tau} \geq c\nu^{-1}(h, H, \tau)\|u^h\|_{A_\tau}, \quad \forall u^h \in V^h.$$

(3) If $c_{H,\tau,h} = \max\{H, H^\gamma\sqrt{H^2/\tau + 1}\}\nu$ is small enough, i.e. $0 < c_{H,\tau,h} \leq c_0$, then there exists a constant $c_p(c_0) > 0$, such that

$$(u^h, P^{D_\tau} u^h)_{A_\tau} \geq c_p \nu^{-1}(h, H, \tau)(u^h, u^h)_{A_\tau}, \quad \forall u^h \in V^h.$$

3.4.2 Proof of the theorem

In this section, we prove the theorem stated in the last section. Again, we need to prove that the assumptions made for the Theorem 1.3 hold and also show how the constants depend on the mesh parameters h, H and τ. The proofs look more complicated because of the lack of overlap between the substructures. But the idea behind the proof is similar. We first establish a bounded decomposition lemma for V^h functions. Note that

the decomposition is no longer uniform in $\|\cdot\|_{D_\tau}$ norm; the bounds depend mildly on the size parameters h, H.

We rewrite Lemma 2.4 in the way that we shall use later in this section.

Lemma 3.11 *Let Ω_i be a substructure, $u^h \in V_i^h$, denote*

$$\overline{u_i^h} = \frac{1}{area(\Omega_i)} \int_{\Omega_i} u^h d\Omega.$$

Then

$$\|u^h - \overline{u_i^h}\|^2_{L^\infty(\Omega_i)} \leq C(1 + \log H/h) \mid u^h \mid^2_{H^1(\Omega_i)}.$$

Lemma 3.12 *For any $u^h \in V^h$, let*

$$u^h = \sum_{(i,j)\in\Lambda^E} u_{ij}^h$$

be the decomposition constructed in Lemma 2.5. We then have

$$\sum_{(i,j)\in\Lambda^E} \|u_{ij}^h\|^2_{L^2(\Omega_{ij})} \leq C(\|u^h\|^2_{L^2(\Omega)} + H^2(1 + \log H/h)\|u^h\|^2_{H_0^1(\Omega)}),$$

where the constant $C > 0$ is independent of H and h.

Proof: For a given $u^h \in V^h$, we begin with the estimate of the $L^2(\Omega)$ norm of $u_{00}^h = I_H u^h$. Let us consider one substructure at a time. Assume that Ω_i has vertices T_1, T_2, T_3 and denote

$$\alpha_i = \frac{1}{\text{area}(\Omega_i)} \int_{\Omega_i} u^h d\Omega.$$

Then, we have

$$\|I_H u^h\|^2_{L^2(\Omega_i)} \leq C(\|I_H u^h - \alpha_i\|^2_{L^2(\Omega_i)} + \|\alpha_i\|^2_{L^2(\Omega_i)}).$$

Since the function $I_H u^h - \alpha_i$ is linear in the region Ω_i, a straightforward calculation shows that the L^2 norm in Ω_i can be bounded by

$$(CH^2(I_H u^h - \alpha_i)^2 \mid_{T_1} + (I_H u^h - \alpha_i)^2 \mid_{T_2} + (I_H u^h - \alpha_i)^2 \mid_{T_3}).$$

Using Lemma 3.11, the above sum is bounded by

$$CH^2(1 + \log H/h) \mid u^h \mid^2_{H^1(\Omega_i)}.$$

To bound the $\|\alpha_i\|_{L^2(\Omega_i)}$ term, we need to estimate the following integral

$$\|\alpha_i\|^2_{L^2(\Omega_i)} \qquad \int_{\Omega_i} (\frac{1}{\text{area}(\Omega_i)} \int_{\Omega_i} u^h d\Omega)^2 d\Omega.$$

Applying Hölder's inequality, this is bounded by

$$\text{area}(\Omega_i)(\frac{1}{\text{area}(\Omega_i)})^2 \int_{\Omega_i} 1 d\Omega \cdot \int_{\Omega_i} (u^h)^2 d\Omega \;=\; \|u^h\|^2_{L^2(\Omega_i)}.$$

Therefore, we obtain

$$\|I_H u^h\|^2_{L^2(\Omega_i)} \leq C(\|u^h\|^2_{L^2(\Omega_i)} + H^2(1 + \log H/h) \mid u^h \mid^2_{H^1(\Omega_i)}).$$

By summing this inequalities for $i = 1, \cdots, N$, and using Poincaré's inequality and replacing the H^1 seminorm by the H^1 norm, we obtain the estimate for u^h_{00} as,

$$\|I_H u^h\|^2_{L^2(\Omega)} \leq C(\|u^h\|^2_{L^2(\Omega)} + H^2(1 + \log H/h)\|u^h\|^2_{H^1_0(\Omega)}).$$

Now consider the case $(i, j) \neq (0, 0)$. Recall the construction,

$$u^h_{ij} \;=\; I_h(\theta_{ij}(u^h - I_H u^h)).$$

Let us denote

$$\Lambda^h_{ij} \;=\; \Omega_{ij} \cap \Lambda^h.$$

By Lemma 3.3, we have

$$\|u_{ij}^h\|_{L^2(\Omega_{ij})}^2 \leq Ch^2 \sum_{x_k \in \Lambda_{ij}^h} ((I_h(\theta_{ij}(u^h - I_H u^h)))(x_k))^2.$$

Since $|\ \theta_{ij}\ | \leq 1$, and x_k is nodal point, the interpolation operator I_h can be removed. Therefore the right hand side can be bounded by

$$Ch^2 \sum_{x_k \in \Lambda_{ij}^h} ((u^h - I_H u^h)(x_k))^2.$$

If we sum over all $(i,j) \in \Lambda^E \backslash (0,0)$, taking into account of the fact that at each nodal point the function value contributes at most three times, then

$$\sum_{(i,j)\in\Lambda^E\backslash(0,0)} \|u_{ij}^h\|_{L^2(\Omega_{ij})}^2 \leq Ch^2 \sum_{x_k \in \Lambda^h} ((u^h - I_H u^h)(x_k))^2.$$

Using Lemma 3.3 again, we can bound the right hand side by

$$C\|u^h - I_H u^h\|_{L^2(\Omega)}^2,$$

which can be bounded by

$$C\|u^h\|_{L^2(\Omega)}^2 + H^2(1 + \log H/h)\|u^h\|_{H_0^1(\Omega)}^2.$$

The proof is completed by adding overall (i,j). □

Lemma 3.13 *For any $u^h \in V^h$, let*

$$u^h = \sum_{(i,j)\in\Lambda^E} u_{ij}^h$$

be the function decomposition constructed in Lemma 2.5, then there exists a constant $C > 0$, independent of H, h, τ, such that

$$\sum_{(i,j)\in\Lambda^E} \|u_{ij}^h\|_{\tau(\Omega_{ij})}^2 \leq C\nu(h,H,\tau)\|u^h\|_{\tau(\Omega)}^2.$$

Proof: The proof is accomplished by combining the Lemma 3.12 and the Lemma 2.5. Since

$$\begin{aligned}\sum_{(i,j)\in\Lambda^E} \|u_{ij}^h\|^2_{\tau(\Omega_{ij})} &\leq C(\|u^h\|^2_{L^2(\Omega)} + H^2(1+\log H/h)\|u^h\|^2_{H_0^1(\Omega)} \\ &\quad +\tau(1+\log H/h)\|u^h\|^2_{H_0^1(\Omega)}).\end{aligned}$$

which is bounded bounded from above by

$$C\nu(h,H,\tau)\|u^h\|^2_{\tau(\Omega)}.$$

□

Lemma 3.14 *There exists a constant $C > 0$, independent of h, H and τ, such that*

$$|\, N_\tau(u^h, P^{D_\tau}u^h)\,| \leq C\max\{H, H^\gamma\sqrt{H^2/\tau+1}\}\nu(h,H,\tau)D_\tau(u^h, P^{D_\tau}u^h), \quad \forall u^h \in V^h.$$

Proof: It is easy to see that the coarse mesh projection $P_{00}^{D_\tau}$ is identical to the coarse mesh projection $P_0^{D_\tau}$ defined in section 2 of this chapter. It follows from the proof of Lemma 3.8 part (1) and (2), that we have

$$|\, N_\tau(P^{D_\tau}u^h, u^h)\,| \leq C\max\{H, H^\gamma\sqrt{H^2/\tau+1}\}\sum_{(i,j)\in\Lambda^E}\|u^h\|^2_{\tau(\Omega_{ij})}. \qquad (3.9)$$

By using the same argument as when we proved equation (3.8) and the decomposition lemma 3.13, we have

$$\|u^h\|^2_{A_\tau} \leq C\nu(h,H,\tau)D_\tau(u^h, P^{D_\tau}u^h). \qquad (3.10)$$

□

The proof of Theorem 3.3 is completed by using Theorem 1.3, the lemmas in this section and the assumption that $\max\{H, H^\gamma\sqrt{H^2/\tau+1}\}\nu(h,H,\tau)$ is small enough.

Chapter 4

Numerical results

4.1 Stationary convection-diffusion problems

4.1.1 The model problems

In this section, we present some model problems that we use to test the additive Schwarz algorithm and the iterative substructuring algorithm for the stationary convection-diffusion equation. We consider the following linear second order elliptic problem defined on $\Omega = [0,1] \times [0,1] \subset R^2$,

$$Lu = -\frac{\partial}{\partial x}(\xi \frac{\partial u}{\partial x}) - \frac{\partial}{\partial y}(\eta \frac{\partial u}{\partial y}) + \alpha \frac{\partial u}{\partial x} + \beta \frac{\partial u}{\partial y} + \gamma u = f,$$

with the homogenous Dirichlet boundary condition. The coefficients are specified as follows.

Example 0. $\xi = 1$, $\eta = 1$ and $\alpha = \beta = \gamma = 0$. This is a selfadjoint problem. We shall use to test the iterative substructuring algorithm. f is chosen so that the solution has the form $u = xe^{xy}sin(\pi x)sin(\pi y)$.

Example 1. $\xi = 1 + x^2 + y^2$, $\eta = e^{xy}$, $\alpha = 5(x + y)$, $\beta = 1/(1 + x + y)$ and $\gamma = 0$. u is the same as in Example 0.

Example 2. The coefficients are chosen as $\xi = \sigma$, $\eta = \sigma$, $\alpha = 1$, $\beta = 1$ and $\gamma = 1$. σ will be specified later. u is the same as in Example 0.

The H-level subdivision of Ω is introduced in a simple way shown in the following picture. We further divide each subregion in the same fashion into h-level triangles, which are not shown in the picture.

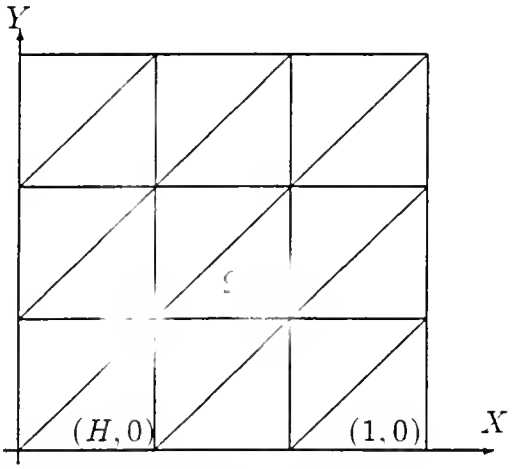

Figure 1. H−level subdivision

Let us introduce some notations. **ite** denotes the total number of iterations required for the GMRES algorithm. **max err** denotes the maximum error between the numerical solution of the Galerkin's equation and the exact solution of the continuous equation. **ovlp** measures the size of the overlap, that is a integer multiple of the fine mesh size h. In our programs, all the subproblems are solved exactly by the band solver from LINPACK. The stopping criterion for the GMRES method is $\| r_i \|_A / \| r_0 \| \leq 10^{-4}$, where r_i is the i^{th} step residual defined in Chapter 1. The programs were run in single precision on a CONVEX C-1 computer at NYU.

4.1.2 Tests of the additive Schwarz method

Based on the 2-level subdivision of Ω, we introduce our basic subregion Ω_i' obtained by extending each triangle Ω_i to a larger triangle such that each

side is parallel to its corresponding side, and does not cut through any h-level triangles. We assume the extension of each side of the Ω_i is uniform in the sense that they all extend the same number of h-level triangles. See the following picture. We cut off anything which is outside Ω.

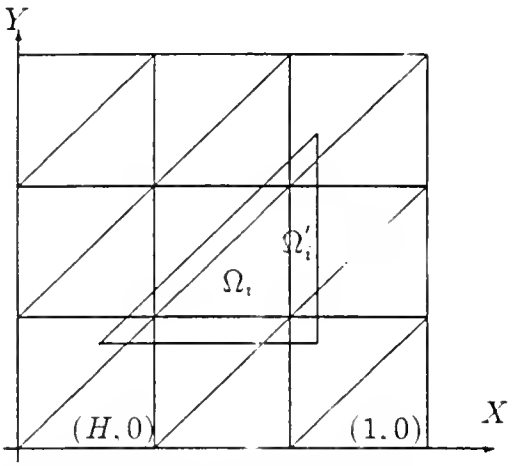

Figure 2. extended subregions

We note that some subregions Ω_i' that intersect with the boundary $\partial\Omega$ are no longer triangles. A lot of regions with different shapes are introduced even if the original Ω_i has a nice shape. This causes difficulties with applying fast solvers in practical applications. There are some ways around it, but more on the theory and the numerical experiments still need to be carried out. For example, we can replace the $\Omega_i's$ by rectangles with approximately the same size, so that the FFT can be used as a subproblem solver.

Example 1. In this example, we plan to test the effectiveness of the algorithm for different mesh sizes h and H with a fixed overlapping factor α. But, since the overlap must set to be a integer multiple of the fine mesh size h, it turns out that we cannot always use the same overlapping factor

for all h and H. In the following table, if $H = 1/3$, we set the overlap to be approximately $0.4H$, and if $H = 1/5$, to be $0.33H$.

h	H	ite	H	ite	max err
1/15	1/3	14	1/5	12	5.0×10^{-3}
1/30	1/3	14	1/5	14	1.3×10^{-3}
1/45	1/3	15	1/5	15	5.8×10^{-4}
1/60	1/3	15	1/5	15	3.0×10^{-4}

For fixed H, the optimality in h can be seen clearly, i.e. **ite** changes very slowly if we use the finer mesh size h. However if we change H, the number of iteration required to achieve the same accuracy does change. That is related to the parameter δ in Theorem 1.3 which is increased if the coarse mesh is further refined subject to the constraint $H \leq H_0$. This can be seen more clearly for problems with higher Reynold's number as in the following examples.

In the theory for additive Schwarz, we assume that the distance between the interior boundary of Ω_i and the boundary of Ω_i' is to be bounded from below by a constant α times the coarse mesh size H. Now, in the following table we show how this constant affects the number of iterations. We use the same model problem as above. Set $H = 1/5$ and $h = 1/60$. We run the same program for different sizes of overlap, which range from one fine grid size to six grid size. The results are shown in the following table.

ovlp	h	$2h$	$3h$	$4h$	$5h$	$6h$
ite	18	18	16	15	14	13

Indeed, increasing the overlap reduces the number of iterations. But we have to note that this also increases the size of the subproblems and the cost per iteration is increased also. Further studies on how to determine the optimal overlap so that the parallel CPU time or the serial CPU time can be minimized are definitely needed. Some discussions about this issue for symmetric problems can be found in [16].

Example 2. (1) $\sigma = \sqrt{2}/30$.

We test the algorithm with a higher Reynold's number. We first take $h = 1/45$ and vary the parameter H. It can be seen from the following table that when $H \leq H_0 = 1/9$ the number of iterations does not depend on H any more, therefore, the algorithm is seen to be optimal. In the following tables the overlapping factor is between 1/3 and 1/4,

H	1/3	1/5	1/9	1/15
ite	+50	37	20	11
ovlp	$5h$	$3h$	$2h$	$1h$

As pointed in Theorem 2.1, the observed parameter H_0 should not change if we change the mesh size parameters. This is demonstrated in the following table.

H=1/10	h=1/80,**ovlp**=$2h$	h=1/100,**ovlp**=$3h$	h=1/120,**ovlp**=$4h$
ite	19	20	20

(2) $\sigma = \sqrt{2}/100$

In this example, we increase the Reynold number to 100.0. It can be seen that by choosing a suitable $H \leq H_0$, we can still control the total number of iterations taken to achieve a specified accuracy. By runing the same example on a coarser fine mesh space an approximate H_0 can be obtained. We refine the mesh so that the required accuracy is achieved.

According to Theorem 2.1 this further refinement of the fine mesh should not increase the number of iterations. A further refinement of the coarse mesh reduces the total number of iterations. See the following tables.

	h=1/45,**ovlp**=1h	h=1/60,**ovlp**=1h	h=1/75,**ovlp**=2h
H=1/15	23	23	23

	h=1/80,**ovlp**=1h	h=1/100,**ovlp**=2h	h=1/120,**ovlp**=2h
H=1/20	15	21	21

4.1.3 Tests of the Iterative substructuring method

Since few numerical experiments of $IS\!M$ for symmetric problems have been done before, we first present some results for the symmetric problems. We then concentrate on the convection-diffusion problems. Compared with the $AS\!M$, the $IS\!M$ is easier to implement because we do not need to determine the overlap size.

We take Ω_i to be the same as for $AS\!M$, and let Ω_{ij} be the union the adjacent pair Ω_i and Ω_j. The subproblems are solved exactly.

Example 0. Since the operator P^B in this example is symmetric with respect to the $(\cdot,\cdot)_A$, see [11], the GMRES method is equivalent to the conjugate gradient method.

Let us first fix H and vary h,

h	1/12	1/24	1/36	1/48	1/60	1/72
H=1/3	12	13	14	14	14	14

In the next set of examples, we fix h and vary H,

H	1/2	1/3	1/4	1/5	1/6	1/10	1/12	1/15	1/20
h=1/60	11	14	15	14	14	13	12	11	10

As predicted by the theory, the number of iteration depends only mildly on the mesh parameters h and H. The restriction on the coarse mesh size does not apply in the symmetric case. We will see later that refinement of the coarse mesh can considerably reduce the number of iterations for non-symmetric problems, however, this is not true for the symmetric problems. As an example, we repeat the first set of tests shown above with a finer coarse mesh,

h	1/12	1/24	1/36	1/48	1/60	1/72
H=1/4	12	14	14	14	15	15

Example 1. This is a nonsymmetric problem. However, the nonsymmetric part is not large compared with the symmetric part. In the following table we show how the number of iterations depend on the mesh parameters.

h	1/15	1/30	1/45	1/60	1/75	1/90	1/105
H=1/3	16	18	19	20	20	20	21
H=1/5	14	17	18	18	18	18	18

We can see clearly that the refinement of the coarse mesh can reduce the total number of iterations, which is not true for the symmetric problems; cf. Theorem 2.2.

Example 2. (1) $\sigma = \sqrt{2}/30$.

For the problems with higher Reynold's number, the effect of the coarse mesh size becomes clearer. We first use $h = 1/45$ to run a series tests for different H. Results are shown in the following table,

H	1/5	1/9	1/15
h=1/45	41	29	18

which tells us that the proper coarse size is about $H = 1/15$. This is

confirmed by the following tests for different h.

H	1/6	1/10	1/12	1/15	1/20
h=1/60	37	23	20	20	13
h=1/120	43	28	24	20	17

(2) $\sigma = \sqrt{2}/100$.

In order to maintain atively small number of iterations for problems with high Reynold's nu we need to select a finer coarse mesh, which depends strongly on the R old's number but not the fine mesh size. This fact suggests we should min e proper coarse mesh by running a series of tests for problem smaller size with the same Reynold's number. Once the coarse mesh size is determined, we can further refine our fine mesh so that the desired accuracy can be obtained. The number of iterations grows mildly as h decreases.

h	1/30	1/45	1/60	1/75	1/90
H=1/15	20	24	30	34	36

h	1/60	/80	1/100	1/120	1/140
H=1/20	20	23	26	28	30

4.2 Parabolic convection-diffusion problems

In this section, we consider the parabolic convection-diffusion equations defined on the unit square $[0,1] \times [0,1] \subset R^2$, which have been discretized by either the backward Euler scheme or the implicit Crank-Nicolson scheme in time and by a Galerkin finite element method in space. We present numerical results for computing the solution at a fixed time level. In the following examples, the elliptic part of the parabolic equation are chosen

to be the same as in section 1 and all the space discretizations that we use in this section are the same as those introduced in the last section.

We assume that the time step has the form

$$\tau = h^{\epsilon},$$

where $\epsilon > 0$ is called the time step tolerance. The main issue here is to determine how the rate of convergence (or number of iterations) depends on ϵ. In the case $\epsilon = 2.0$, the stiffness matrix is well conditioned in the sense that the condition number does not depend on the mesh parameters h and τ. In general, the smaller ϵ is, i.e. the larger the time step is, the more ill-conditioned is the problem. If ϵ is too small, the accuracy of the solution is completely lost. We do include some test results using unusually large time step τ, compared with the space mesh size h, to show the robustness of the algorithm. In our experiments, we assume that $0.25 \leq \epsilon \leq 1.5$.

4.2.1 Tests of the additive Schwarz method

	ϵ	1.5	1.0	0.5	0.25
Example 0	h=1/45,H=1/3,ovlp=5h	10	12	12	13
	h=1/60,H=1/3,ovlp=6h	10	12	13	13
	h=1/60,H=1/10,ovlp=2h	10	11	12	12
	h=1/45,H=1/3,ovlp=h	15	18	18	19
Example 1	h=1/[illegible] H=1/3,ovlp=2h	12	13	14	14
	h=1/[illegible] H=1/3,ovlp=4h	12	14	14	14
	h=1/[illegible] H=1/3,ovlp=6h	12	14	15	15
	h=1/60,H=1/3,ovlp=8h	11	14	15	15
	h=1/[illegible] H=1[illegible] lp=1h	11	12	12	12
	h=1/[illegible] H=1/5,ovlp=2h	12	14	14	14
	h=1/45,H=1/5,ovlp=3h	12	13	14	15
	h=1/60,H=1/5,ovlp=4h	11	13	15	15
Example 2 $\sigma = \sqrt{2}/30$	h=1/45,H=1/3,ovlp=5h	11	17	28	35
	h=1/45,H=1/9,ovlp=2h	9	12	14	16
	h=1/45,H=1/15,ovlp=1h	7	8	10	10
	h=1/60,H=1/10,ovlp=2h	9	11	14	15
	h=1/60,H=1/15,ovlp=1h	7	9	10	11
	h=1/60,H=1/20,ovlp=1h	6	8	10	10
Example 2 $\sigma = \sqrt{2}/100$	h=1/60,H=1/15,ovlp=1h	8	11	18	20
	h=1/60,H=1/20,ovlp=1h	7	9	13	16

This table shows that the number of iterations almost does not depend on the mesh parameters as well as the time step size, especially for symmetric problems. For nonsymmetric, see Theorem 3.1. For nonsymmetric problems, a relatively small coarse mesh size is needed for reducing the total number of iterations, that was reflected in Theorem 3.1 part(3). The larger the Reynold's number is, the smaller the coarse mesh size is needed. The same as in the stationary case, using less overlap can increase the total number of iterations. This can be seen from the last row of Example 0.

4.2.2 Tests of the modified additive Schwarz method

	ϵ	1.5	1.0	0.5	0.25
Example 0	h=1/45,H=1/3,ovlp=5h	10	12	14	15
	h=1/60,H=1/3,ovlp=6h	9	12	15	16
	h=1/60,H=1/10,ovlp=2h	12	24	34	35
	h=1/45,H=1/3,ovlp=h	12	21	27	28
Example 1	h=1/15,H=1/3,ovlp=2h	13	15	15	15
	h=1/30,H=1/3,ovlp=4h	13	14	16	16
	h=1/45,H=1/3,ovlp=6h	13	15	16	17
	h=1/60,H=1/3,ovlp=8h	12	15	17	17
	h=1/15,H=1/5,ovlp=3h	14	18	19	20
	h=1/30,H=1/5,ovlp=6h	15	20	23	24
	h=1/45,H=1/5,ovlp=9h	13	19	24	25
	h=1/60,H=1/5,ovlp=12h	13	18	23	25

	ϵ	1.5	1.0	0.75	0.5
Example 2 $\sigma = \sqrt{2}/30$	h=1/45,H=1/3,ovlp=5h	13	22	31	42
	h=1/45,H=1/5,ovlp=3h	12	18	23	32
	h=1/45,H=1/9,ovlp=2h	10	14	19	30
	h=1/45,H=1/15,ovlp=1h	7	12	22	36
	h=1/60,H=1/10,ovlp=2h	10	13	20	33
	h=1/60,H=1/15,ovlp=1h	8	13	24	41
Example 2 $\sigma = \sqrt{2}/100$	h=1/45,H=1/9,ovlp=2h	10	19	31	+50
	h=1/45,H=1/15,ovlp=1h	7	13	25	48
	h=1/60,H=1/15,ovlp=1h	7	13	25	+50
	h=1/60,H=1/20,ovlp=1h	6	12	24	+50

This table shows that if the factor τ/H^2 is small, the results are quite satisfactory compared with the corresponding AS_M using coarse mesh. This is ture especially for symmetric and mildly nonsymmetric problems; cf. Example 0 and Example 1. However, if τ/H^2 is large, i.e. the time step or the number of substructures is large, the number of iterations can be large; cf. the third row of Example 0. For problems with high Reynold's number,

the algorithm becomes very sensitive to the time step parameter ϵ; cf. the last column of the second table. See also Theorem 3.2.

We also note that using less overlap can increase the total number of iterations. This can be seen by comparing the first and last rows in Example 0.

By comparing with t[illegible] ASM, we see that this modified algorithm is more sensitive to the m[illegible] arameters and the overlapping factor.

4.2.3 Tests of the iterat[illegible] substructuring method

	ϵ	1.5	1.0	0.5	0.25
Example 0	h=1/45,H=1/3	14	14	14	14
	h=1/60,H=1/3	15	15	14	14
	h=1/60,H=1/10	11	12	12	13
Example 1	h=1/45,H=1/5	18	17	17	17
	h=1/45,H=1/9	14	14	14	14
	h=1/60,H=1/5	18	17	17	18
	h=1/60,H=1/10	13	14	14	14
Example 2 $\sigma = \sqrt{2}/30$	h=1/45,H=1/5	15	25	35	38
	h=1/45,H=1/9	13	18	20	22
	h=1/45,H=1/15	10	12	13	14
	h=1/60,H=1/5	15	22	36	41
	h=1/60,H=1/10	13	17	20	22
	h=1/60,H=1/15	11	14	15	16
Example 2 $\sigma = \sqrt{2}/100$	h=1/45,H=1/15	11	15	20	22
	h=1/60,H=1/15	12	18	24	27
	h=1/60,H=1/20	10	14	17	18

For the symmetric problems, we see that the number of iterations is independent of the fine mesh size h, but depends on the coarse mesh size H. The number of iterations is reduced when the coarse mesh size is reduced. This is not true for time independent problems. The algorithm is not very sensitive to the time step parameter.

For nonsymmetric problems, the algorithm becomes more sensitive to the time step parameter and the coarse mesh size is more important for controlling the number of iterations.

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